First Year Analysis Exam, September 2008

Answer each question on a separate sheet of paper. Write all solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let X and Y be metric spaces. Prove, if $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

2. Let X, Y be metric spaces with distance functions d_X and d_Y respectively. Prove, that $d : X \times Y \to [0, \infty)$ defined by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

is a metric on $X \times Y$.

Suppose $f : X \to Y$ and X is compact. Prove f is continuous if and only if the graph of f,

$$G(f) = \{(x, f(x)) : x \in X\} \subset X \times Y,$$

is compact.

3. Let X be a metric space and $a \in X$. Suppose there is an open connected set containing a. Prove there is a largest open connected set containing a; i.e., there is an open connected set $W \ni a$ such that that if $V \ni a$ is any other open connected subset of X, then $V \subset W$.

4. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable. Show if f' is Riemann integrable, then

$$f(b) - f(a) = \int_a^b f'(t)dt.$$

5. Let $(p_n)_n$ be a sequence in a metric space X. Show, if there is a point $p \in X$ such that every subsequence of $(p_n)_n$ has a further subsequence which converges to p, then $(p_n)_n$ converges to p.

6. Suppose $f : [-1, 1] \to \mathbb{R}$ is continuous. If

$$\int_{-1}^{1} f(x)x^n dx = 0$$

for all $n = 0, 1, \ldots$, what can you say about f?

If instead, the integrals above are 0 only for odd n, what can you say about f?

7. Assume $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies

- for each t the function $F_t : [0,1] \to \mathbb{R}$ given by $F_t(x) = K(x,t)$ is (Lebesgue) measurable;
- for each x, the function $G_x : [0,1] \to \mathbb{R}$ given by $G_x(t) = K(x,t)$ is continuous; and
- there is a Lebesgue integrable function h so that $|K(x,t)| \leq h(x)$ for all $(x,t) \in [0,1] \times [0,1]$.

Show that $\Phi : [0,1] \to \mathbb{R}$ given by

$$\Phi(t) = \int_0^1 K(x,t)dx$$

is continuous.

8. Let X be a set and Σ a σ-algebra of subsets of X. Suppose $h_n : X \to \mathbb{R}$ is a uniformly bounded sequence of Σ-measurable functions and let

$$h = \limsup h_n.$$

Prove, for each $a \in \mathbb{R}$, the set $h^{-1}([a,\infty)) \in \Sigma$.

9. Give examples, if possible, of the following. Briefly justify your answers.

- A differentiable function f on $\mathbb{R}$ whose derivative is continuous, except for a jump discontinuity at 0.
- A set of positive Lebesgue measure which contains no interval.
- A differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) > 0$, but f is not increasing on any open interval containing 0.
- A sequence of functions $(f_n)_n$ in $L^2(\mu)$ which converges in $L^2(\mu)$, but not pointwise a.e. μ.