First Year Analysis Exam, May 2008

Answer each question on a separate sheet of paper. Write all solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose (a_n) and (b_n) are bounded sequences of real numbers. Prove,

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Give an example in which strict inequality holds.

2. Fix $\alpha > 0$ and let X, Y be metric spaces. A function $f : X \to Y$ is *α continuous* if for each $x \in X$ there is a δ so that if $s, t \in X$ and both $d_X(s,x), d_X(t,x) < \delta$, then $d_Y(f(s), f(t)) < \alpha$. The function f is *uniformly α continuous* if it is possible to choose δ independently of $x \in X$. Prove, if f is α continuous and X is compact, then f is uniformly α continuous.

3. Let X be a metric space. Prove, if $A, B \subset X$ are connected and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

4. Suppose $f : [a,b] \to \mathbb{R}$ is differentiable. Show if f' is Riemann integrable, then

$$f(b) - f(a) = \int_a^b f'(t)dt.$$

5. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle in the complex plane. Prove, if $f : \mathbb{T} \to \mathbb{C}$ is continuous and

$$\int_0^1 f(e^{it})e^{-int}dt = 0$$

for all $n \in \mathbb{Z}$, then $f = 0$.

6. Let (X, Σ, μ) be a measurable space. Prove, if $f : X \to \mathbb{R}$ is integrable, then for each ϵ there is a δ so that if $A \in \Sigma$ and $\mu(A) < \delta$, then

$$|\int_A f d\mu| < \epsilon.$$

7. Assume $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ satisfies

- for each t the function $F_t : [0,1] \longrightarrow \mathbb{R}$ given by $F_t(x) = K(x,t)$ is (Lebesgue) measurable;
- for each x, the function $G_x : [0,1] \longrightarrow \mathbb{R}$ given by $G_x(t) = K(x,t)$ is continuous; and
- there is a Lebesgue integrable function h so that $|K(x,t)| \leq h(x)$ for all $(x,t) \in [0,1] \times [0,1]$.

Show that $\Phi : [0,1] \longrightarrow \mathbb{R}$ given by

$$\Phi(t) = \int_0^1 K(x,t)dx$$

is continuous.

8. Let X a set, Σ a σ-algebra of subsets of X and $f : X \longrightarrow \mathbb{R}$ be given. Prove, if for each $a \in \mathbb{R}$ the set $f^{-1}((a,\infty)) \in \Sigma$, then $f^{-1}(V) \in \Sigma$ for each Borel set $V \subset \mathbb{R}$.

9. Give examples of the following, with brief justification, if possible.

- A sequence of nonempty closed sets $C_1 \supset C_2 \supset C_3 \supset \ldots$ such that $\cap C_n = \emptyset$. Is this possible if instead the C_n are assumed compact (and of course still nonempty and nested decreasing)?
- A bounded function $f : [0,1] \longrightarrow \mathbb{R}$ which is not Riemann integrable.
- A sequence of continuous functions which converges pointwise to a continuous function but for which the convergence is not uniform.
- A sequence of functions $(f_n)_n$ in $L^2(\mu)$ which converges in $L^2(\mu)$, but not pointwise a.e.