First-Year Analysis Examination
January 2008

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let $\{K_\alpha\}$ be a collection of compact subsets of a metric space X, with the property that every finite subcollection has nonempty intersection. Prove that $\cap_\alpha K_\alpha$ is nonempty.

2. Suppose f and f' are defined in a neighborhood of a point $a \in \mathbb{R}$, and $f''(a)$ exists. Prove that

$$\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a).$$

   Show by example that the limit can exist even if $f''(a)$ does not.

3. Let $f : [a,b] \to \mathbb{R}$ and define the graph $G(f)$ of f by

$$G(f) = \{(x, f(x)) : x \in [a,b]\}.$$

   Prove that f is continuous on $[a,b]$ if and only if $G(f)$ is a compact subset of $\mathbb{R}^2$.

4. Let $C[a,b]$ denote the metric space of continuous real-valued functions on $[a,b]$ with the metric

$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

   Define $L : C[a,b] \to \mathbb{R}$ by $L(f) = \int_a^b f(x)\,dx$. Prove that L is continuous.

5. Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

6. Suppose that $\{f_n\}$ and $\{g_n\}$ are sequences of bounded real-valued functions that converge uniformly on a set E. Prove $\{f_n g_n\}$ converges uniformly on E. Give an example to show that the conclusion can be false if the functions are not bounded.

7. Let $\{f_n\}$ be a sequence of continuous real-valued functions converging uniformly on a compact set E. Prove that the set $\{f_n\}$ is equicontinuous on E.

8. Let $\{f_n\}$ be a sequence of measurable functions and suppose $f_n \to f$ pointwise. Prove that f is measurable.

9. For each of the following, prove or give a counterexample (m denotes Lebesgue measure on $\mathbb{R}$; assume all functions are measurable):

   a) If f_n is integrable for all n, $f_n \to f$ uniformly on $\mathbb{R}$ and f is integrable, then $\int_{\mathbb{R}} f_n \, dm \to \int_{\mathbb{R}} f \, dm$.

   b) If $f_n \to f$ uniformly on $[0, 1]$ and f is integrable, then $\int_0^1 f_n \, dm \to \int_0^1 f \, dm$.

   c) Suppose $f_n \geq 0$ and $\int_0^1 f_n \, dm = 1$ for all n. If $f_n \to f$ pointwise then $\int_0^1 f \, dm \leq 1$, and the inequality may be strict.

10. Let $f \in L^1(\mathbb{R})$ and define

$$F(x) = \int_{-\infty}^{x} f(t) \, dt.$$

Prove that F is continuous.