First-Year Analysis Examination
August 2007

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Prove that compact subsets of metric spaces are closed.

2. Prove that for any two real sequences $\{a_n\}, \{b_n\}$,

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

   provided the right hand side is not of the form $\infty - \infty$.

3. Let (X, d) be a complete metric space. Suppose $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) < \frac{1}{2^n}$ for all n. Prove that $\{x_n\}$ converges to some $x \in X$.

4. Let E be a nonempty subset of a metric space (X, d).

   a) Define the distance of a point $x \in X$ to E to be

$$\rho_E(x) = \inf_{y \in E} d(x, y).$$

   Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$, the closure of E.

   b) Prove that ρ_E is uniformly continuous on X.

5. Suppose that X and Y are metric spaces and $f : X \to Y$ is uniformly continuous. Prove that $\{f(x_n)\}$ is a Cauchy sequence in Y whenever $\{x_n\}$ is a Cauchy sequence in X.

6. Let $\{s_n\}$ be a sequence of complex numbers. Define

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k$$

   Prove that if $s_n \to s$, then $\sigma_n \to s$. Give an example of a divergent sequence s_n for which $\sigma_n \to 0$.

7. Let $\{f_n\}$ be an equicontinuous sequence of functions on a compact metric space X. Prove that if $\{f_n\}$ converges pointwise on X then $\{f_n\}$ converges uniformly.

8. Suppose that f is differentiable on $\mathbb{R}$ and $f'(x) \to 0$ as $x \to +\infty$. Prove that

$$\lim_{x \to +\infty} [f(x+1) - f(x)] = 0.$$

9. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on a measure space $(X, \mathscr{M}, \mu)$ which decreases pointwise to a function f. Suppose that $\int f_k \, d\mu < +\infty$ for some k. Prove that

$$\lim \int f_n \, d\mu = \int f \, d\mu.$$

Give an example to show that the hypothesis $\int f_k \, d\mu < +\infty$ is necessary.

10. Suppose that $|f(x,y)| \leq 1$ for all $x, y \in [0,1]$, and that $f(x,y)$ is continuous in y for each fixed x, and continuous in x for each fixed y. Is

$$g(x) = \int_0^1 f(x,y) \, dy$$

necessarily continuous?