First-Year Analysis Examination
September 2006

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose that S_1 and S_2 are nonempty bounded subsets of $\mathbb{R}$. Prove that

$$\text{lub}\{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\} = \text{lub}S_1 + \text{lub}S_2$$

where lub denotes least upper bound.

2. Let f be a continuous, one-to-one mapping of a compact metric space X onto a metric space Y. Prove that f^{-1} is a continuous mapping from Y to X. Show by example that if compactness is not assumed then the inverse may not be continuous.

3. Let $(s_n)_{n=1}^{\infty}$ be a sequence of complex numbers and let $t_n = (s_1 + \cdots + s_n)/n$. Prove that $s_n \to s$ implies $t_n \to s$. Show that the converse is false, by a suitable example.

4. Suppose that f is a differentiable real-valued function on $[a, b]$ and suppose that $f'(a) < \lambda < f'(b)$. Prove that there exists a point x in (a, b) such that $f'(x) = \lambda$.

5. Let E be a compact metric space and let (f_n) be a sequence of continuous real-valued functions on E converging pointwise to a function f on E; suppose that $f_n(x) \geqslant f_{n+1}(x)$ for all $x \in E$ and all n. Prove that if f is continuous then $f_n \to f$ uniformly on E and show by example that convergence may not be uniform when f is not continuous.

6. Let (f_n) be an equicontinuous sequence of real-valued functions on $[a, b]$. Prove that if $f_n \to f$ pointwise then $f_n \to f$ uniformly.

7. Let $C[0, 1]$ be the space of all continuous real-valued functions on $[0, 1]$ with the metric d defined by $d(f, g) = \max\{|f(x) - g(x)| : x \in [0, 1]\}$. Prove that the closed unit ball $B = \{f \in C[0, 1] : d(f, \phi) \leqslant 1\}$ is not compact, where $\phi(x) = 0$ for all $x \in [0, 1]$. (*Hint*: consider $f_n(x) = x^n$.)

8. Let f be a Lebesgue-integrable function on $[a, b]$ and define $F(x) = \int_a^x f(t)dt$ for $a \leqslant x \leqslant b$. Prove that F is continuous on $[a, b]$.

9. Let C be a closed subset of $[a, b]$. Prove that there exists a sequence (g_n) of continuous functions on $[a, b]$ such that $\lim_{n\to\infty} \int_a^b |g_n(x) - K_C(x)|^2 dx = 0$ where K_C is the characteristic function of C.

10. Let f be a nonnegative measurable function and E a measurable set. Prove that if $\int_E f d\mu = 0$ then $f = 0$ almost everywhere on E.