First-Year Analysis Examination
May 2006

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose f is a continuous one-to-one mapping of a metric space X onto a metric space Y.

(a) Prove that if X is compact, then the inverse mapping f^{-1} is continuous.

(b) Give an example to show that f^{-1} may not be continuous if X is not compact.

2. Let X be the set of all continuous, real-valued functions defined on the interval $[0,1]$.

(a) Show that a metric d is defined on X by the rule that if $f, g \in X$ then

$$d(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

(b) Let A denote the set of $f \in X$ such that $f(t) > 0$ for all $t \in [0,1]$. Is A an open subset of X?

3. Suppose that $a_n > 0$ and that the series $\sum a_n$ diverges. Prove that the series $\sum \frac{a_n}{1+a_n}$ also diverges.

4. Let $[a,b]$ be a compact interval in $\mathbb{R}$ and let f be a real-valued function differentiable on $[a,b]$. Assume that f' is continuous and satisfies $|f'(t)| < 1$ at each $t \in [a,b]$. Prove that there exists $c \in (0,1)$ such that if $x, y \in [a,b]$ then

$$|f(x) - f(y)| \le c|x-y|.$$

5. Let E be a compact interval on the real line. Prove that the family of all polynomials of degree at most N with coefficients in $[-1,1]$ is uniformly bounded and equicontinuous on E.

6. (a) Let $h : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, and let V be an open subset of $\mathbb{R}$. Prove that $h^{-1}(V)$ is a measurable subset of $\mathbb{R}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Prove that $f \circ g$ is Lebesgue measurable.

7. Let m be Lebesgue measure on the real line $\mathbb{R}$ and let E be a measurable subset of $\mathbb{R}$. Prove that for any $\varepsilon > 0$ there exists an open subset G of $\mathbb{R}$ such that $E \subset G$ and $m(G - E) < \varepsilon$.

8. Let X be a measurable space with non-negative measure μ and let f be a measurable function on X such that $\int_X |f|^p d\mu < \infty$ for some $p > 0$. Prove that

$$\mu\{x \in X : |f(x)| \geq s\} \leq s^{-p} \int_X |f|^p d\mu.$$

9. Suppose $f \in \mathcal{L}(\mu)$. Prove that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\int_E f d\mu| < \varepsilon$ for all measurable sets E with $\mu(E) < \delta$.

10. Prove that

$$\int_0^1 x^{-x} dx = \sum_1^{\infty} n^{-n}.$$

Hint: Consider integrating a suitable series term-by-term.