First-Year Analysis Examination
January 2006

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let d be a metric on the set M and define $D(x, y) = d(x, y)/(1 + d(x, y))$ when $x, y \in M$. Prove that: (a) D is a *bounded* metric on M; (b) D and d determine precisely the same open sets in M.

2. Suppose that K and F are disjoint sets in a metric space X, K is compact and F is closed. Prove that there exists a $\delta > 0$ such that $d(p, q) \geqslant \delta$ if $p \in K$, $q \in F$. Show by example that the conclusion may fail for two disjoint closed sets if neither is compact.

3. Let the function $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \sqrt{x}$ for $x \geqslant 0$. Prove that f is uniformly continuous on its domain.

4. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges. Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geqslant 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum(a_n/s_n)$ diverges.

5. Let f be a continuous, real-valued function defined on $(-1, 1)$. Suppose that for all $x \in (-1, 1)$ with $x \neq 0$, $f'(x)$ exists. Suppose also that $\lim_{x\to 0} f'(x) = L$, for some real number L. Does it follow that $f'(0)$ exists and equals L? Prove your answer.

6. Prove directly (without reference to Lebesgue theory) that a monotonic real-valued function on $[a, b]$ is Riemann integrable over $[a, b]$.

7. Suppose that K is a compact metric space and $\{f_n\}$ is a sequence of real-valued continuous functions on K that converges uniformly on K. Prove that $\{f_n\}$ is equicontinuous on K.

8. Does the sequence $\{x^n(1 - x^n)\}$ converge uniformly for x in the interval $[0, 1]$? Prove your assertion.

9. Let f be a square-integrable function on $\mathbb{R}$ (that is $\int_{\mathbb{R}} |f|^2 dm < \infty$, where m denotes Lebesgue measure). Prove that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| dm < \varepsilon$ whenever E is a Lebesgue-measurable subset of $\mathbb{R}$ with $m(E) < \delta$.

10. Let f be a measurable function on the measure space E. Prove that if $\int_A f d\mu = 0$ for each measurable subset A of E then $f = 0$ almost everywhere.