First-Year Analysis Examination
January 2005

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Show that the set

$$A = \{x \in \mathbb{R} : x \leqslant a_n \text{ infinitely often}\}$$

is nonempty and bounded above. Without *assuming* the convergence of real Cauchy sequences, show that $a_n \to \sup A$ as $n \to \infty$.

2. Let $(a_n)_{n=0}^{\infty}$ be a sequence of non-negative real numbers. Show that if the series $\sum_{n\geqslant 0} a_n$ converges then the series $\sum_{n\geqslant 0} \sqrt{a_n a_{n+1}}$ converges and show by example that the converse may fail.

3. Let M be a metric space and for each integer $n \geqslant 0$ let $A_n \subset M$. For each of the following pairs of sets, decide which (if any) is always a subset of the other; in each case, decide whether inclusion can be proper.
(i) $\overline{\cap_{n\geqslant 0} A_n}$ and $\cap_{n\geqslant 0}\overline{A_n}$;
(ii) $\overline{\cup_{n\geqslant 0} A_n}$ and $\cup_{n\geqslant 0}\overline{A_n}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $|f(y) - f(x)| \geqslant k|y - x|$ for all $x, y \in \mathbb{R}$ and some fixed $k > 0$. Prove carefully that f is surjective.

Suggestion: It might help to show first that f is injective and has closed image.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous, with the property that $f(x, y) < 0$ when $xy > 0$ and $f(x, y) > 0$ when $xy < 0$. Show that the initial value problem

$$\begin{cases} \phi'(t) = f(t, \phi(t)) \\ \phi(0) = 0 \end{cases}$$

has a unique (differentiable!) solution $\phi : \mathbb{R} \to \mathbb{R}$.

Suggestion: Consider the extreme values of ϕ on $[0, T]$ for $T > 0$.

6. Let $f : (a, \infty) \to \mathbb{R}$ be differentiable and let A and B be real numbers. Prove that if $f(x) \to A$ and $f'(x) \to B$ as $x \to \infty$ then $B = 0$.

7. Let $(f_n : n \geqslant 0)$ be a sequence of continuous real-valued functions on the metric space M decreasing pointwise to 0. Prove Dini's theorem that if M is compact then the convergence is uniform.

8. Let $(f_n : n \geqslant 0)$ be a pointwise-bounded sequence of measurable functions on the measurable space E. Show that the set $A = \{x \in E : \lim f_n(x) \text{ exists}\}$ is measurable.

9. Let μ be a positive measure on the measurable space E and assume that $\mu(E) < \infty$. Show that if $(f_n : n \geqslant 0)$ is a sequence of bounded measurable functions on E and $f_n \to f$ uniformly on E then

$$\lim \int_E f_n d\mu = \int_E f d\mu.$$

Show that the assumption $\mu(E) < \infty$ cannot be removed.

10. Let μ be a positive measure on the measurable space E. Prove that if $f \geqslant 0$ on E and $\int_E f d\mu = 0$ then $f = 0$ almost everywhere on E.