First-Year Analysis Examination
September 2004

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Recall that the real numbers $\mathbb{R}$ form an ordered field that is complete in the sense that it enjoys the least upper bound property.
(i) Show that $\mathbb{R}$ has the Archimedean property: if $a, b \in \mathbb{R}$ and $a > 0$ then there exists an integer $n > 0$ such that $na > b$.
(ii) Show by example that an ordered field having the Archimedean property may lack the least upper bound property.

2. (i) Let $(a_n : n \in \mathbb{N})$ be a real *sequence* converging to the real number a. Prove that if π is a permutation of $\mathbb{N}$ then $a_{\pi(n)} \to a$ as $n \to \infty$.

(ii) Show by example that rearrangements of a convergent real *series* need not converge.

3. Let $f : (a, b) \to \mathbb{R}$ be continuous, bounded and monotonic. Prove carefully that f is uniformly continuous.
Suggestion: Consider $f(a+)$.

4. Let M be a compact metric space. Prove that each sequence in M has a convergent subsequence. Hence, or otherwise, prove that M is complete.

5. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove or give a counterexample.
(i) If f' is bounded, then f is uniformly continuous.
(ii) If f is uniformly continuous, then f' is bounded.

6. Suppose $K : [0, 1] \times [0, 1] \to \mathbb{R}$ is a bounded function, measurable in each variable separately, such that there is a constant C so that if $0 \leq x, y \leq 1$, then

$$\int_0^1 |K(x,t) - K(y,t)|\, dt \leq C|x - y|.$$

Suppose $(f_n : n \in \mathbb{N})$ is a uniformly bounded sequence of Lebesgue integrable functions from $[0, 1]$ to $\mathbb{R}$. Define $F_n : [0, 1] \to \mathbb{R}$ by

$$F_n(x) = \int_0^1 K(x,t) f_n(t) dt.$$

Does the sequence $(F_n : n \in \mathbb{N})$ have a uniformly convergent subsequence?

7. Fix $a < c < b$. Show, if $\alpha : [a,b] \to \mathbb{R}$ is increasing and continuous at c and if $f : [a,b] \to \mathbb{R}$ is bounded and continuous except possibly at c, then f is Riemann integrable with respect to α.

Give an example which shows that the continuity of α at c is necessary.

8. Let (X, Σ, μ) denote a measure space. Show, if f is integrable with respect to μ, then for every $\epsilon > 0$ there is a $\delta > 0$ so that if $A \in \Sigma$ and $\mu(A) < \delta$, then

$$\Big| \int_A f d\mu \Big| < \epsilon.$$

9. Suppose $g_n : [0, 2\pi] \to \mathbb{C}$ is a uniformly bounded sequence of Lebesgue integrable functions which converges pointwise. Let

$$f_n(z) = \frac{1}{2i\pi} \int_0^{2\pi} \frac{g_n(t)}{\exp(it) - z} dt, \quad |z| < 1.$$

Does the sequence f_n converges pointwise for $|z| < 1$?

10. Give examples of the following, if possible. You need not prove your answer, but do give a brief justification.

(i) A bounded function $f : [0,1] \to \mathbb{R}$ which is not Riemann integrable.

(ii) An $f \in C(\mathbb{T})$ which is not in the uniform closure of the set

$$\{p(\gamma) = \sum_{j=0}^{n} p_j \gamma^j : n \in \mathbb{N}, p_j \in \mathbb{C}\}.$$

Here $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, $C(\mathbb{T})$ denotes the continuous complex-valued functions on $\mathbb{T}$, and $\gamma \in \mathbb{T}$.

(iii) A differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) > 0$, but which is not increasing on any open interval containing 0.

(iv) A sequence of functions which converges in $L^1([0,1])$, but not in $L^2([0,1])$.