First-Year Analysis Examination
May 2004

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let $\{d_n\}_{n=1}^{\infty}$ be a sequence of positive reals and assume that the series $\sum_{n=1}^{\infty} d_n$ is divergent. Show carefully, if the sequence $\{d_n\}_{n=1}^{\infty}$ converges to 0, then the series

$$\sum_{n=1}^{\infty} \frac{d_n}{1+d_n^2}$$

diverges. Give an example to show that the hypothesis $\{d_n\}_{n=1}^{\infty}$ converges to 0 is necessary.

2. Let X be a metric space and $f : X \to \mathbb{R}$ a real-valued function. Prove that the following conditions are equivalent:

(i) for each $b \in \mathbb{R}$ the set $f^{-1}(-\infty, b) \subset X$ is open;

(ii) for every $x \in X$ and every sequence $\{x_n\}$ converging to x,

$$f(x) \geqslant \inf\{f(x_n) : n \geqslant 1\}.$$

3. State the (open cover) definition of compact subset of a metric space and use the definition to prove, if K is a compact subset of a metric space X, then K is closed.

4. Suppose $f : X \to Y$ is a uniformly continuous mapping from the metric space X to the metric space Y.

(i) Show, if $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.

(ii) Show, if, in addition to the uniform continuity of f, that f is a bijection, f^{-1} is continuous, and Y is complete, then X is complete.

5. State precisely and prove a version the fundamental theorem of calculus whose conclusion is

$$F(b) - F(a) = \int_a^b F'(t)dt.$$

6. Suppose $\{f_n\}$ is a uniformly bounded sequence of real valued Lebesgue integrable functions on the interval $[0,1]$ and define $F_n : [0,1] \to \mathbb{R}$ by

$$F_n(x) = \int_0^x f_n(t)dt.$$

Prove there is a subsequence of the sequence $\{F_n\}$ which converges uniformly.

7. Let (X, Σ, μ) denote a measure space which is complete in the sense that if $A \in \Sigma$, $B \subset A$, and $\mu(A) = 0$, then $B \in \Sigma$. Given $f, g : X \to \mathbb{R}$, show if f is measurable, and $E = \{x \in X : f(x) \neq g(x)\}$ is a set of measure 0 (so $E \in \Sigma$ and $\mu(E) = 0$), then g is measurable.

8. Suppose

(a) $f : [0,1] \times [0,1] \to [0,1]$;

(b) for each $0 \leq x \leq 1$, the function $g_x : [0,1] \to [0,1]$ defined by $g_x(y) = f(x,y)$ is Lebesgue measurable; and

(c) for each $0 \leq y \leq 1$, the function $h_y : [0,1] \to [0,1]$ defined by $h_y(x) = f(x,y)$ is continuous.

Explain why it is possible to define $\Phi : [0,1] \to \mathbb{R}$ by

$$\Phi(x) = \int_0^1 f(x,y)dy.$$

Is Φ continuous?

9. Let (X, Σ, μ) denote a measure space and suppose $\mu(X) < \infty$. Suppose that $\{\phi_n\}_{n \in \mathbb{Z}}$ is an $L^2(\mu)$ orthonormal sequence of real valued functions. Show, if $E \in \Sigma$, $\mu(E) > 0$, and $\delta > 0$, then there are at most finitely many integers n so that $\phi_n(x) \geq \delta$ for all $x \in E$.

10. Give examples of the following, if possible. You need not prove your answer, but do give a brief justification.

(i) A bounded function $f : [0,1] \to \mathbb{R}$ which is not Riemann integrable.

(ii) A closed and bounded, but not compact, subset C of a complete metric space X.

(iii) A differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) > 0$, but which is not increasing on any open interval containing 0.

(iv) A sequence of functions which converges in $L^2([0,1])$, but which does not converge pointwise almost everywhere.