First-Year Analysis Examination
January 2004

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) < 0 < f(b)$. By considering the set

$$\{t \in [a, b] : f(t) < 0\}$$

or otherwise, prove carefully that there exists $p \in (a, b)$ such that $f(p) = 0$.

Warning: The Intermediate Value Theorem may not be used unless it is proved.

2. Let $f : X \to Y$ be a map between metric spaces and let $(x_n : n \geqslant 0)$ be a Cauchy sequence in X.

(i) Show that if f is uniformly continuous then $(f(x_n) : n \geqslant 0)$ is a Cauchy sequence in Y.

(ii) Show by example that $(f(x_n) : n \geqslant 0)$ need not be Cauchy if f is merely continuous.

3. Let $(a_n : n \geqslant 0)$ be a sequence of strictly positive real numbers. Show that *if* the series $\sum_{n \geqslant 0} a_n$ converges *then* the series $\sum_{n \geqslant 0} \sqrt{a_n a_{n+1}}$ converges. Show also that the converse is true *provided* the sequence $(a_n : n \geqslant 0)$ is monotonic.

4. Let A be a given *non*-compact subset of $\mathbb{R}$. Prove that there exist:

(i) a continuous function $f : A \to \mathbb{R}$ that is not bounded;

(ii) a continuous function $g : A \to \mathbb{R}$ that is bounded but does not attain its bounds.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying the initial value problem (IVP)

$$f' = f \text{ and } f(0) = 1.$$

(i) Show that if $s, t \in \mathbb{R}$ then $f(s - t)f(t) = f(s)$.

(ii) Deduce that f is everywhere strictly positive.

(iii) Conclude that the given IVP has at most one solution.

Warning: The exponential function may *not* be assumed.

6. Give examples of the following, if possible.

(i) A measure space (X, Σ, μ) and sequence of integrable functions $\{f_n\}$ which converges to an integrable function f uniformly, but for which the sequence $\{\int_X f_n d\,\mu\}$ does not converge to $\int_X f d\,\mu$.

(ii) A bounded function $f : [0,1] \to \mathbb{R}$ which is not Riemann integrable.

(iii) A closed bounded subset of $C([0,1])$ which is not compact.

7. Let (X, Σ, μ) be a measure space. Let $\{g_n\}$ be a sequence of integrable functions which converges almost everywhere to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions such that

$$|f_n| \leq g_n$$

and so that $\{f_n\}$ converges to a function f almost everywhere. Show, if

$$\lim \int g_n d\,\mu = \int g d\,\mu,$$

then

$$\lim \int f_n d\,\mu = \int f d\,\mu.$$

Suggestion: Consider the proof of the dominated convergence theorem.

8. Suppose $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is continuous. Show, if $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of (Lebesgue) measurable functions on the interval $[0,1]$, then the sequence $\{F_n\}_{n=1}^{\infty}$ defined by

$$F_n(x) = \int_0^1 K(x,t) f_n(t) dt$$

is equicontinuous on $[0,1]$. Must $\{F_n\}$ have a uniformly convergent subsequence?

9. Define $\alpha : [-1,1] \to \mathbb{R}$ by $\alpha(x) = x$ for $-1 \leq x \leq 0$ and $\alpha(x) = 1 + x$ for $0 < x \leq 1$. Explain why $f : [-1,1] \to \mathbb{R}$ given by $f(x) = \exp(x)$ is Riemann-Stieltjes integrable on $[-1,1]$ with respect to α and compute the integral.

10. Let $\mathcal{R}$ be a σ-ring on a set X which is not a σ-algebra (so $X \notin \mathcal{R}$). Set $\mathcal{R}' = \{X \setminus E : E \in \mathcal{R}\}$ and show $\mathcal{B} = \mathcal{R} \cup \mathcal{R}'$ is the smallest σ-algebra containing $\mathcal{R}$; i.e., $\mathcal{B}$ is the σ-algebra generated by the collection $\mathcal{R}$.