First-Year Analysis Examination
September 2003

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let the real line $\mathbb{R}$ be the disjoint union of nonempty sets A and B such that if $a \in A$ and $b \in B$ then $a < b$. Prove carefully that there exists a unique $p \in \mathbb{R}$ such that $(-\infty, p) \subset A$ and $(p, \infty) \subset B$.

2. Let A be a subset of the metric space M.
   (i) Prove that if A is complete, then A is closed.
   (ii) Prove that if M is complete and A is closed, then A is complete.

3. Let $(a_n : n \geqslant 1)$ be a bounded sequence of real numbers and for $n \geqslant 1$ define

$$\sigma_n = \frac{1}{n}(a_1 + \cdots + a_n).$$

Show that

$$\liminf a_n \leqslant \liminf \sigma_n \leqslant \limsup \sigma_n \leqslant \limsup a_n$$

whence if $(a_n : n \geqslant 1)$ converges then so does $(\sigma_n : n \geqslant 1)$ to the same limit.

4. Let M be a compact metric space and $f : M \to \mathbb{R}$ an upper semicontinuous function. Prove that
   (i) f is bounded above on M;
   (ii) f achieves its supremum on M.
   [By definition, f is upper semicontinuous precisely when $f^{-1}(-\infty, a)$ is open in M for each $a \in \mathbb{R}$.]

5. State the Mean Value Theorem.
   Let the real-valued function f be continuous on $[a, b)$ and differentiable on (a, b). Show that if

$$\lim_{x \downarrow a} f'(x) = L$$

then also

$$\lim_{x \downarrow a} \frac{f(x) - f(a)}{x - a} = L.$$

6. Show, if $f : [0,1] \to \mathbb{R}$ is Riemann integrable, then $F : [0,1] \to \mathbb{R}$ defined by

$$F(x) = \int_0^x f(t)dt$$

is differentiable at points of continuity of f.

7. Show, if $f : [0,1] \times [0,1] \to \mathbb{C}$ is continuous and $\epsilon > 0$, then there exists a polynomial p in two variables,

$$p(x,y) = \sum_{m,n=0}^{N} p_{m,n} x^m y^n \quad (p_{m,n} \in \mathbb{C})$$

such that $|f(x,y) - p(x,y)| < \epsilon$ for all $0 \le x, y \le 1$.

8. Suppose f is Lebesgue integrable on $\mathbb{R}$. Show, if

$$\int_a^b f dx = 0$$

for all $a \le b \in \mathbb{R}$, then

$$\int_A f dx = 0$$

for all Borel sets A in $\mathbb{R}$.

9. Suppose f is Lebesgue integrable on $[a,b]$. Show $F : [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt$$

is continuous.

10. Give (with brief justification) examples of the following, if possible.

(i) A function that is in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$.

(ii) A sequence of Lebesgue integrable functions $f_n : \mathbb{R} \to \mathbb{R}$ such that f_n converges pointwise to a Lebesgue integrable function f, but such that the sequence $\{\int_{\mathbb{R}} f_n dx\}$ does not converge to $\int_{\mathbb{R}} f dx$.

(iii) A sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ that converges pointwise to a continuous function $f : [0,1] \to \mathbb{R}$, but so that the sequence $\{f_n\}$ does not converge to f in the metric space $C([0,1])$. Does your answer change if the sequence is equicontinuous?