First-Year Analysis Examination
May 2003

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Prove that if $(a_n : n \geq 0)$ and $(b_n : n \geq 0)$ are bounded sequences of real numbers then

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty}(a_n) + \liminf_{n\to\infty}(b_n).$$

Show by example that the inequality can be strict.

2. Let $(a_n : n \geq 0)$ be a sequence of positive real numbers and assume that the series $\sum_{n=0}^{\infty} a_n$ diverges.
(a) Does the series $\sum_{n=0}^{\infty} \frac{a_n}{1+a_n}$ converge or diverge? Explain.
(b) Does the series $\sum_{n=0}^{\infty} \frac{a_n}{1+n^2 a_n}$ converge or diverge? Explain.

3. Let M be a metric space and $f : M \to \mathbb{R}$ a continuous function. Show that if M is compact then f is necessarily bounded. Does boundedness of f follow from precompactness of M? Explain.

[Recall that by definition, M is *precompact* when for each $\epsilon > 0$ there exists a cover of M by finitely many ϵ-balls.]

4. State the Intermediate Value Theorem.
Let n be a positive integer and let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1)$. Show that there exist points $a, b \in [0,1]$ such that $b - a = \frac{1}{n}$ and $f(a) = f(b)$.

[It might help to consider $g(t) = f(t) - f(t - \frac{1}{n})$ for t in a suitable interval, along with the particular sum $g(\frac{1}{n}) + g(\frac{2}{n}) + \cdots g(1)$.]

5. State the Mean Value Theorem.
Let $f : (a,b) \to \mathbb{R}$ be continuous everywhere and differentiable except perhaps at $p \in (a,b)$. Show that if $\lim_{t\to p} f'(t)$ exists and equals the real number L, then f is differentiable at p and $f'(p) = L$. Is it necessary to assume that f be continuous everywhere? Explain.

6. Let $F : [a,b] \to \mathbb{R}$. Show, if F is differentiable on $[a,b]$ and F' is Riemann integrable on $[a,b]$, then

$$F(b) - F(a) = \int_a^b F'(x)dx.$$

7. Suppose $\{f_n\}$ is a uniformly bounded sequence of Lebesgue integrable functions on the interval $[0,1]$. Let

$$F_n(x) = \int_0^x f_n(t)dt.$$

Does the sequence $\{F_n\}$ have a uniformly convergent subsequence?

8. Let $\mathcal{M}$ denote the Lebesgue measurable sets in the real line $\mathbb{R}$ and suppose $\mu : \mathcal{M} \to \mathbb{R}$ is a (countably additive) measure. Show that if μ is regular and $\mu((-\infty, t)) = 0$ for all $t \in \mathbb{R}$ then $\mu = 0$.

[Recall that μ *regular* means that for each $A \in \mathcal{M}$ there exist Borel sets F and G with $F \subset A \subset G$ and $\mu(G \setminus A) = 0 = \mu(A \setminus F)$.]

9. Give (with brief justification) examples of the following.
(a) A sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ which converges pointwise to a continuous function $f : [0,1] \to \mathbb{R}$, but where the convergence is not uniform.
(b) A bounded function which is not Riemann integrable.
(c) A ring $\mathcal{R}$ and a countably additive set function $\phi : \mathcal{R} \to [0, \infty)$ which cannot be extended to a countably additive set function on a σ-ring that contains $\mathcal{R}$.

10. Let (X, Σ, μ) denote a measure space. Suppose $F : (0,1) \times X \to \mathbb{R}$ satisfies each of the following conditions:
(a) for each $t \in (0,1)$ the function $f_t : X \to \mathbb{R}$ given by $f_t(x) = F(t,x)$ is measurable;
(b) for each $x \in X$ the function $g_x : (0,1) \to \mathbb{R}$ given by $g_x(t) = F(t,x)$ is continuous;
(c) there exists a μ-integrable function $G : X \to [0, \infty)$ with the property that $|F(t,x)| \le G(x)$ for all t and x.

Show that

$$\Phi(t) = \int_X F(t,x)d\mu(x)$$

is defined and continuous on $(0,1)$.