

First-Year Analysis Examination January 2003

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose X and Y are metric spaces. Prove, if X is compact and $f : X \rightarrow Y$ is continuous, then $f(X)$ is compact.

2. State a theorem which generalizes the alternating series test and use it to determine those $z \in \mathbb{C}$ for which the series

$$\sum_{n=2}^{\infty} \frac{z^n}{\sqrt{n}}$$

converges.

3. Give an example of a function which is continuous, but not uniformly continuous. Prove, if X and Y are metric spaces, $f : X \rightarrow Y$ is uniformly continuous, and $\{x_n\}$ is a Cauchy sequence in X , then $\{f(x_n)\}$ is a Cauchy sequence in Y .

4. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at every point of \mathbb{R} and is differentiable except possibly at 0. If

$$\lim_{x \rightarrow 0} f'(x) = L$$

(that is the limit exists as a finite number L), does it follow that f is differentiable at 0?

5. Define $\alpha : [-1, 1] \rightarrow \mathbb{R}$ by $\alpha(x) = 0$ if $x \leq 0$ and $\alpha(x) = 1$ if $x > 0$. Show, a bounded function $f : [-1, 1] \rightarrow \mathbb{R}$ is Riemann integrable with respect to α if and only if $f(0+) = f(0)$.

6. Suppose $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous. Show, if $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of (Lebesgue) measurable functions on the interval $[0, 1]$, then the sequence $\{F_n\}_{n=1}^{\infty}$ defined by

$$F_n(x) = \int_0^1 K(x, t) f_n(t) dt$$

is equicontinuous on $[0, 1]$. Must $\{F_n\}$ have a uniformly convergent subsequence?

7. Let L^2 denote $L^2(\mu)$, where μ is Lebesgue measure on $[-\pi, \pi]$. Suppose $\{\phi_n\}_{n=0}^\infty$ is an orthonormal sequence from L^2 and there is a C such that $|\phi_n(x)| \leq C$ for all $n \in \mathbb{N}$ and $x \in [-\pi, \pi]$. Show $\{\phi_n\}$ does not converge pointwise (in the interval $[-\pi, \pi]$).

Prove that the sequence of functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $f_n(x) = \sin(nx)$ does not converge pointwise.

8. Let (X, Σ, μ) be a measure space. Show, if $\mu(X) < \infty$ and if $\{f_n\}$ is a sequence of integrable functions which converges uniformly to a function f , then f is integrable and

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0.$$

Does the conclusion hold if the hypothesis $\mu(X) < \infty$ is omitted?

9. State Fatou's Lemma and use it to prove the Dominated Convergence Theorem.