First-Year Analysis Examination
May 2002

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose X and Y are metric spaces. Prove, if X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.

2. Show, if $\{a_n\}$ and $\{b_n\}$ are bounded sequences of real numbers, then

$$\overline{\lim}(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n.$$

3. Show, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist (as finite numbers), then f is uniformly continuous.

4. State the Mean Value Theorem. A fixed point of $f : \mathbb{R} \to \mathbb{R}$ is a point $x \in \mathbb{R}$ such that $f(x) = x$. Prove, if f is differentiable and $f'(t) \neq 1$ for all $t \in \mathbb{R}$, then f has at most one fixed point.

5. Suppose $\alpha : [0, 1] \to \mathbb{R}$ is strictly increasing. If $f : [0, 1] \to \mathbb{R}$ is continuous, $f \geq 0$, and

$$\int_0^1 f d\alpha = 0,$$

what can be said about f?

6. Suppose $K : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. Show, if $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of (Lebesgue) measurable functions on the interval $[0, 1]$, then the sequence $\{F_n\}_{n=1}^{\infty}$ defined by

$$F_n(x) = \int_0^1 K(x, t) f_n(t) dt$$

is equicontinuous on $[0, 1]$. Must $\{F_n\}$ have a uniformly convergent subsequence?

7. Let (X, Σ, μ) be a measure space and $\{f_n\}$ a sequence of measurable real-valued functions defined on X. Show the set of points x such that the sequence $\{f_n(x)\}$ converges is measurable.

8. Suppose $f : [0,1] \times [0,1] \to [0,1]$. Show, if

(a) For each $0 \le x \le 1$, $g_x(t) = f(x,t)$ is continuous; and

(b) For each $0 \le t \le 1$, $h_t(x) = f(x,t)$ is (Lebesgue) measurable,

then

$$\Phi(t) = \int_0^1 f(x,t)dx$$

is defined and continuous on $[0,1]$.

9. State the Monotone Convergence Theorem and use it to prove Fatou's Lemma.