First-Year Analysis Examination
January 2002

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. State the definition of a convergent series. Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\log n}$ converges.

2. Let $f_n(x) = \sin nx$ for $n = 1, 2, \ldots$ and $x \in [0, 2\pi]$. Prove that $\{f_n\}$ does not contain a subsequence which converges pointwise on $[0, 2\pi]$.

3. State the definition of a compact space. Let $f : X \to Y$ be continuous, where X and Y are metric spaces. Prove that $f(X)$ is compact if X is compact.

4. Suppose that f is a uniformly continuous mapping of a metric space X into a metric space Y. Prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

5. Let $f_n(x) = nx^2/(1 + nx)$ for $x \in [0, 1]$.
   (A) Compute $\lim_{n\to\infty} f_n(x)$.
   (B) Is the convergence uniform? Prove your assertion.
State the definition of uniform convergence. State the theorem used in your proof.

6. Suppose f' is continuous on $[a, b]$. Let $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that

$$|(f(t) - f(x))/(t - x) \; - \; f'(x)| < \varepsilon$$

whenever $0 < |t - x| < \delta$ and $a \le x, t \le b$.

7. Let A be a dense subset of a metric space. Suppose U is an open set. Prove that $U \subset \overline{(A \cap U)}$, where $\overline{(A \cap U)}$ is the closure of $A \cap U$.

8. State and prove Fatou's lemma. Show that the inequality may be strict.

9. Let f be Lebesgue integrable on $\mathbb{R}$ and suppose that $\int_I f\,dx = 0$ for every interval I. Prove that $f = 0$ a.e..