First-Year Analysis Examination
September 2001

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose $a_n \geqslant 0$, for $n = 1, 2, \ldots$, and $\sum a_n$ converges. Does $\sum a_n/\sqrt{n}$ converge? Prove.

2. State and prove the Mean Value Theorem.

3. Let A be a dense subset of a metric space, suppose U is an open set. Prove that $U \subset \overline{(A \cap U)}$, where $\overline{(A \cap U)}$ is the closure of $A \cap U$.

4. Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a compact metric space X. Suppose $\{f_n\}$ converges pointwise to a continuous function f on X and $f_n(x) \geqslant f_{n+1}(x)$ for all $x \in X$ and $n = 1, 2, \ldots$. Prove that $\{f_n\}$ converges uniformly on X.

5. Let (X, d) be a complete metric space, $f : X \to X$ and assume that there exists a $k \in (0, 1)$ such that $d(f(x), f(y)) \leqslant k\, d(x, y)$ for all $x, y \in X$.

(a) Prove $d(x_{n+p}, x_n) \leqslant (k^{p-1} + \cdots + 1)\, k^n\, d(x_1, x_0) \leqslant \frac{k^n}{1-k}\, d(x_1, x_0)$ where $x_{n+1} = f(x_n)$ for $n = 1, 2, \ldots$.

(b) Show that there exists exactly one $\overline{x} \in X$ such that $f(\overline{x}) = \overline{x}$.
Hint: Is $\{x_n\}$ a Cauchy sequence?

6. State and prove Fatou's lemma. Show the inequality may be strict.

7. If f is μ-integrable and $\int_A f\, d\mu = 0$ for every measurable subset A, what can you say about f? Prove your answer.

8. If $\{f_n\}$ is a sequence of Lebesgue integrable functions on $\mathbb{R}$ such that $f_n \to 0$ uniformly on $\mathbb{R}$, does it follow that $\int_{\mathbb{R}} f_n\, dm \to 0$? Prove or disprove. (m denotes Lebesgue measure.)

9. Let $\{f_n\}$ be a sequence of measurable functions. Prove that the set of points x at which $\{f_n(x)\}$ converges is a measurable set.