First Year Examination in Analysis May, 2001

Be sure to write your name on each sheet turned in. Give reasons for each step used in your proof. All work must be presented in a neat and logical fashion.

① Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Assume that $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = 0$. Prove that f is uniformly continuous on $\mathbb{R}$.

② Let $f(x) = x^2 \sin(1/x)$, $x \neq 0$; $f(0) = 0$.

(a) Compute $f'(x)$ for all x.

(b) Is f' continuous at $x = 0$? Prove.

③ Let $f : X \to Y$ be continuous, where X and Y are metric spaces. Prove $f(X)$ is compact if X is compact.

④ Let F be a closed subset of $\mathbb{R}$. Prove that F is a countable union of compact sets.

⑤ Define $\alpha(x) = 0$ if $x \leq 0$ and $\alpha(x) = 1$ if $x > 0$. Let $f : [-1, 1] \to \mathbb{R}$ be bounded. Prove that $f \in \mathcal{R}(\alpha)$ if and only if $f(0+) = f(0)$.

⑥ (a) Define the exponential function $E(z)$.

(b) Prove that $E(z+w) = E(z)\,E(w)$.

(7) Let $f:\mathbb{R}\to\mathbb{R}$ be Lebesgue integrable. Suppose $\int_A f\,dm = 0$ for every measurable set A.
What can you say about f? Prove your answer.

(8) State and prove the Lebesgue dominated convergence theorem.

(9) Let $E \subset [0,1]$. Suppose E is a Lebesgue measurable set. Let $0 \le \alpha \le m(E)$. Prove that there exits a measurable subset F of E such that $m(F)=\alpha$.
[Hint: Examine the function $f(x) = m([0,x]\cap E)$, $x\in[0,1]$]

(10) Suppose $f:[a,b]\to\mathbb{R}$, $-\infty < a < b < +\infty$, and assume f' is continuous on $[a,b]$. Let $\varepsilon>0$. Prove that there exists a $\delta>0$ such that

$$\left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| < \varepsilon, \text{ whenever } 0<|t-x|<\delta, \quad a\le x\le b,\ a\le t\le b.$$