First Year Exam in Analysis Jan, 2000

All work must be done in a neat and logical fashion. Give reasons for your steps, do not leave gaps in your arguments. Put your name on each sheet of paper and start each problem on a new sheet. There are 9 problems.

1. Let f be continuous on the open interval (a, b). Suppose $f(a+0)$ and $f(b-0)$ exist. Prove that f is uniformly continuous on (a, b).

2. Let f be differentiable in (λ, ∞). Suppose $f'(x) \to A$ as $x \to \infty$. Show that $\frac{f(x)}{x} \to A$ as $x \to \infty$.

3. Define $a_1 = 1$, and $a_{n+1} = \sqrt{2a_n}$. Prove that $\lim a_n$ exists and find this limit.

4. Let g be a continuous function on $[0,1]$, with $g(1)=0$. Prove that $\{x^n g(x)\}_{n=1}^{\infty}$ converges uniformly on $[0,1]$.

5. Prove or disprove that every closed subset of the real line is a countable union of compact sets.

6. Let f be a real valued measurable function. If E is a Borel set, prove $f^{-1}(E)$ is measurable.

7. Let E be a compact set and suppose $\{f_n\}$ is a sequence of continuous functions on E. If $f_{n+1} \le f_n$ on E and $f(x) = \lim f_n(x)$ is continuous, prove that the convergence is uniform on E. Show by an example that the compactness of E is essential.

(8) Suppose $\{f_n\}$ is a sequence of measurable functions on the measurable set E.
Let $A = \{x \in E : \lim_n f_n(x) \text{ exists}\}$. Prove that A is measurable.

(9) Let f be a non-negative integrable function. Prove that for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\int_A f \, d\mu < \varepsilon$, whenever $\mu(A) < \delta$.

[Hint: First prove for simple functions].