First Year Exam in Analysis   Sept. 2000

Write all your steps in a neat and logical fashion, giving reasons for them. Do not leave gaps in your proofs. Each problem should be solved on a separate sheet of paper. Put you name (PRINT) on each sheet. there are 9 problems.

1. Let X be a metric space on which every infinite subset has a limit point. Prove that for any given $\delta > 0$, X can be covered by finitely many neighborhoods of radius δ.

2. Let $f_n(x) = \sin nx$, $n = 1, 2, \ldots$, $x \in [0, 2\pi]$.
   Prove that $\{f_n\}$ does <u>not</u> contain a subsequence which converges pointwise on $[0, 2\pi]$.

3. Let X be a metric space and $f : X \to \mathbb{R}$. We say that f is <u>upper semicontinuous</u> if $f^{-1}((-\infty, a))$ is open in X for every $a \in \mathbb{R}$. Suppose X is compact. Prove that
   (a) $M = \sup_{x \in X} f(x) < \infty$.
   (b) there exists an $x_0 \in X$ such that $f(x_0) = M$.

4. Find all the positive values of b for which the series $\sum_{1}^{\infty} b^{\log n}$ converges

5. Let $\sum a_n x^n$ be a power series. Suppose $\sum a_n x_0^n$ converges. Prove that $\sum |a_n x^n|$ converges for all $|x| < |x_0|$. (Hint: $a_n x_0^n \to 0$).

6. State the theorem for differentiablity of inverse functions. Use this theorem to prove $\arctan x$ is differentiable and compute its derivative

(7) Let $f: [a,b] \to \mathbb{R}$ be a continuous function.
Assume $f \geq 0$ on $[a,b]$ and f is not constant on $[a,b]$, where $a < b$. Prove that $\int_a^b f(x)dx > 0$.

(8) Let (X, Σ, μ) be a measure space, with $\mu(X) < \infty$.
Let (f_n) be a sequence from $L^1(\mu)$ which converges uniformly on X to a function f.
Prove that $f_n \to f$ in the mean and $\int f_n d\mu \to \int f d\mu$.

(9) Let f be an integrable function for (X, Σ, μ), a measure space. Suppose $\int_A f d\mu = 0$ for every $A \in \Sigma$.
Prove that $f = 0$ a.e. μ on X.