First year Analysis exam . May 2000

Note. Write complete computations and explanations. Do not use arrows or other unnecessary signs; use words for explanations.
Write complete sentences.
Each problem should be solved on a separate sheet of paper.

① Let (x_n) be a sequence of real numbers and $x_0 \geq -\infty$ such that $x_n \to x_0$ and $x_n > x_0$ for every n. Prove that one can change the order of the terms of the sequence (x_n) to obtain a decreasing sequence (y_n). What can be said about the sequence (y_n)? Justify your answer.
State the definition of $\lim x_n = -\infty$, with neighborhoods and with ε.

② Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $x_0 \in \overline{\mathbb{R}}$. Assume that for every decreasing sequence (x_n) of real numbers with $x_n > x_0$ and $x_n \to x_0$, the limit $\lim f(x_n)$ exists (finite or infinite).
a.) Show that the limit $L = \lim f(x_n)$ is independent of the sequence (x_n)
b.) Show that $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) = L$.

State the definition of $\lim\limits_{x \to -\infty} f(x) = +\infty$, with neighborhoods and with ε, δ.

③ Let (a_n) be a sequence of real numbers $\neq 0$ and assume there is a number $q > 1$ such that

$$q = \lim \frac{\log\left(\frac{1}{|a_n|}\right)}{\log n}.$$

Prove that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
Hint: Use a real number p such that $1 < p < q$.

(4) Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ a monotone function such that its range $f(A)$ is an interval. Prove that f is continuous on A.
Use this result to prove the continuity of inverse functions.

(5) State the theorem of differentiability of inverse functions.
Prove that the function arctan is differentiable and find its derivative.

(6) Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ a function and $x_0 \in I$. Assume that f is continuous on I and differentiable on the set $I - \{x_0\}$ and that the limit $\lim_{x \to x_0} f'(x_0)$ exists, finite or infinite. Prove that f has a derivative at x_0 and $f'(x_0) = \lim_{x \to x_0} f'(x)$.
Give a complete statement of the theorem used in the proof.

(7) Let $f : (0, +\infty) \to \mathbb{R}$ be a differentiable function.
Suppose that $\lim_{x \to +\infty} [f(x) + f'(x)] = A$, finite
Prove that $\lim_{x \to +\infty} f(x) = A$ and $\lim_{x \to +\infty} f'(x) = 0$
Hint. Use l'Hospital rule for $\dfrac{e^x f(x)}{e^x}$.
State l'Hospital theorem.

(8) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Assume $f \geq 0$ on $[a, b]$ and f is not constant on $[a, b]$, and $a < b$. Prove that $\int_a^b f(x)dx > 0$.

(9) Let (X, Σ, μ) be a measure space. Prove that the set of Σ-step functions of $L^1(\mu)$ is dense in $L^1(\mu)$.
Give a complete statement of the theorems used in the proof.

(10) Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$.
Let (f_n) be a sequence from $L^1(\mu)$, converging uniformly to a function $f : X \to \mathbb{R}$. Prove that $f \in L^1(\mu)$ and $f_n \to f$ in the mean and $\int f_n d\mu \to \int f d\mu$.