Be sure to write your proofs in a neat and logical fashion to receive credit. Do not leave gaps in your proofs. Give reasons for your steps.

1. In a metric space, prove that a compact set is closed.

2. Suppose that f is a uniformly continuous mapping of a metric space X into a metric space Y. Prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

3. Suppose f' is continuous on $[a,b]$. Let $\varepsilon>0$. Prove that there exists a $\delta>0$ such that
$$\left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| < \varepsilon$$
whenever $0<|t-x|<\delta$, $a \le x \le b$, $a \le t \le b$.

4. Let K be a compact subset of a metric space. Suppose $\{f_n\}$ is a uniformly convergent sequence of continuous real valued functions defined on K. Prove that $\{f_n\}$ is equicontinuous.

5. Let $f_n(x) = \dfrac{nx^2}{1+nx}$, $x \in [0,1]$.

   (a) Compute $\lim_n f_n(x)$.

   (b) Is the convergence uniform? Prove.

⑥ Suppose f is continuous on $[a,b]$ and $\int_a^b f(x)\varphi(x)dx = 0$ for every integrable function φ on $[a,b]$. Prove $f \equiv 0$ on $[a,b]$.

⑦ Let (X, Σ, μ) be a measure space and $\{f_n\}$ a sequence of μ-integrable functions such that $f_n \to f$ pointwise. Assume that $\liminf_n \int |f_n| d\mu < \infty$. Prove that f is μ-integrable.

⑧ Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence from $L^1(\mu)$ converging uniformly to a function $f: X \to \mathbb{R}$. Prove that $f \in L^1(\mu)$, $f_n \to f$ in the mean and $\int f_n d\mu \to \int f d\mu$.

⑨ Let (X, Σ) be a measurable space. Prove that a Σ-measurable function $f: X \to [0,\infty)$ is the limit of an increasing sequence of Σ-step functions $\{f_n\}$. Show that if f is bounded, then $f_n \to f$ uniformly on X.