First Year Exam in Analysis Sept. 1999

Write all proofs in a neat and logical fashion. Do not leave any gaps in your proofs. Put each problem on a separate page.

1. Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Prove that the inverse mapping f^{-1} is a continuous mapping of Y onto X.

2. State and prove the Mean Value Theorem.

3. Give the construction to show that every open set in $\mathbb{R}^1$ is the union of at most a countable collection of disjoint open intervals.

4. Suppose K and F are disjoint sets in a metric space, K is compact and F is closed. Prove that there exists a $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$ and $q \in F$. Show that the conclusion may fail for two disjoint closed sets if neither is compact.

5. For any sequence $\{s_n\}$, define $t_n = \dfrac{s_1 + \cdots + s_n}{n}$.

   (a) Prove $s_n \to s$ implies $t_n \to s$.

   (b) Show that there are divergent sequences $\{s_n\}$ which in this manner give rise to a convergent sequence $\{t_n\}$.

6. (a) Define e^z in terms of a power series.

   (b) Prove $e^{z_1+z_2} = e^{z_1}e^{z_2}$

   (There are important steps in which theorems must be used for justification of these steps. Be sure to state them).

7. Let (X, Σ) be a measurable space. Let f be a non negative bounded measurable function. Prove that there exists a sequence of simple measurable functions f_n such that: (1) (f_n) is a monotone sequence;
   (2) (f_n) converges uniformly to f.

8. Let E be a Lebesgue measurable subset of $[a,b]$ and suppose α is a number such that $0 \leq \alpha \leq m(E)$. Prove that there exists a measurable set $F \subset E$ such that $m(F) = \alpha$.

9. Let (X, Σ, μ) be a measure space and suppose f is an integrable function such that $\int_E f d\mu = 0$ for each $E \in \Sigma$. What can you say about f? Prove your answer.