First Year Analysis Exam
May 10, 1999

Give reasons for all your steps. Write all proofs in a logical neat fashion. Turn problems in a separate sheets. Do all problems.

1. Let X and Y be metric spaces. Suppose X is compact and $f : X \longrightarrow Y$ is continuous. Show that $f(X)$ is compact.

2. Let (X,d) be a compact metric space. Let $\{x_n\}$ be a Cauchy sequence of elements in X. Does $\{x_n\}$ converge to an element in X? Prove or disprove.

3. Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put $g(x) = f(x)/x \quad (x > 0)$.

Prove g is monotonically increasing.

4. (a) State and prove the nth root test for series.

(b) Does the series

$$\sum_{n=1}^{\infty} (1 - 1/n)^{n^2} \quad \text{converge?}$$

5. Let K be a compact set in a metric space. Prove that if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then $\{f_n\}$ is uniformly bounded on K.

6. Let

$$f_n(x) = x/(1 + nx^2) \quad \text{for } n = 1,2,3,\ldots .$$

Show that $\{f_n\}$ converges uniformly to a function f. Determine the set on the real line in which

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

7. Let f be a Riemann integrable over the interval [a,b] and define

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F'(x) = f(x)$ a.e. on [a,b].

8. For $0 \leq x \leq 1$, let

$f(x) = 0$ if x is irrational

$1/n$ if $x = m/n$, where m and n do not have common factors.

Is f Riemann inetgrable? If so, find the value of the integral on [0,1].

9. Let E be a measurable subset of [0,1] and $m(E) > 0$. Prove that there exist a pair of points x and y in E such that $x - y$ is a non-zero rational number.

10. Let $\{f_n\}$ be a decreasing sequence of integrable functions on the measurable space X. If

$$\int_X f_n\,d\mu \geq M > -\infty$$

for all n = 1,2,3,... , then

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X \lim_{n\to\infty} f_n\,d\mu .$$