First Year Exam in Analysis Sept, 1998

Be sure to put each problem on a separate sheet. Do not leave any gaps in your profs. State all therems you use in the course of your proof.

1. Let (x_n) be a sequence of real numbers such that $\lim x_n = x$. Suppose that $x_n > x$ for each n. Prove that one can change the order of the terms in (x_n) to obtain a decreasing sequence.
[Hint: Let $\alpha > 0$. How many x_n are outside the interval $(x, x+\alpha)$?]

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2. Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$. Assume that for every decreasing sequence (x_n) with $x_n \downarrow x$ and $x_n > x_0$, the sequence $(f(x_n))$ is Cauchy.
Prove: (a) The limit $l = \lim f(x_n)$ is independent of the sequence.
(b) $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) = l$.

[Hint: Given two such sequences (x_n) and (y_n), form a certain subsequence $z_n \downarrow x_0$ so that z_{2n} is a subsequence of (x_n) and (z_{2n+1}) is a subsequence of (y_n).]

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3. Prove: (1) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
(2) Let (x_n) be a bounded sequence in $\mathbb{R}$. Suppose all convergent subsequences have the same limit point. Prove $\lim x_n$ exists.

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4. Prove that the continuous image of a connected set is connected.

(5) Suppose $a_n > 0$ and $b_n > 0$, $n = 1, 2, \ldots$. Suppose $\sum a_n < \infty$ and $\sum b_n = \infty$. Prove or disprove $\sum \frac{a_n}{b_n} < \infty$.

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(6) Suppose f is differentiable on $(0, \infty)$ and $\lim_{x\to\infty} (f + f') = A$. Prove $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} f'(x) = 0$.

[Hint: Use L'Hopital on $\frac{h(x)}{g(x)}$ where $h(x) = e^x f(x)$. You determine what g should be and supply all details.]

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(7) Let $f: \mathbb{R} \to \mathbb{R}$ be a bounded Lebesgue measurable function. Prove or disprove that there exists a sequence of simple measurable functions which converge uniformly to f.

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(8) State and prove Fatou's Theorem and show that the inequality may be strict.

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(9) Let (f_n) be a sequence of Lebesgue integrable functions defined on $\mathbb{R}$ such that $f_n(x) \to 0$ uniformly on $\mathbb{R}$. Prove or disprove that $\lim_n \int_{\mathbb{R}} f_n \, dm = 0$, where m denotes Lebesgue measure.