First Year Exam in Analysis May, 1998

Put each problem on a separate page. Justify all your steps. Present all work in a neat and logical fashion.

① Let $f : (0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n}, \text{ with } \gcd(m,n) = 1. \end{cases}$$

Find all points $x \in (0,1]$ such that $f$ is continuous at $x$.

② Let $A \subset \mathbb{R}$. Call a point $p \in \mathbb{R}$ a condensation point of $A$ if every neighborhood of $p$ contains uncountably many points of $A$. Let $P = \{q \in \mathbb{R} : q \text{ is a condensation point of } A\}$.

Prove that $P$ is a closed set.

③ Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a compact metric space $K$. Suppose $\{f_n\}$ converges pointwise to a continuous function $f$ on $K$ and $f_n(x) \geq f_{n+1}(x)$, for all $x \in K$ and $n = 1, 2, \ldots$.

Prove that $\{f_n\}$ converges uniformly to $f$ on $K$.

④ If $c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = 0$, where $c_0, \ldots, c_n$ are real constants, prove that the equation $c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n = 0$ has at least one real root between $0$ and $1$.

⑤ Suppose $\sum a_n$ is a convergent series, where $a_n \geq 0$ for each $n$. Discuss the convergence of $\sum \frac{\sqrt{a_n}}{n}$.

⑥ Let $f$ be Lebesgue integrable on $\mathbb{R}$ and suppose $\int_A f \, dm = 0$ for every measurable subset $A \subset \mathbb{R}$.

What can you conclude about $f$?

(7) Let $f$ be a real valued Lebesgue integrable function on $[0,\infty)$ such that $\int_0^t f\,dm \geq 0$ for all $t \geq 0$. Is it true that $f \geq 0$ a.e. Prove your answer.

(8) Let $E \subset [0,1]$ be a measurable set and suppose $0 \leq \alpha \leq m(E)$. Prove that there exists a measurable set $F \subset E$ such that $m(F) = \alpha$. (Hint: Examine the function on $[0,1]$, $f(x) = m([0,x] \cap E)$.)

(9) Let $(X,d)$ be a complete metric space, $f: X \to X$ and assume that there exists a $0 < K < 1$ such that $d(f(x), f(y)) \leq K d(x,y)$ for all $(x,y) \in X \times X$.

(a) Prove $d(x_{m+p}, x_m) \leq (K^{p-1} + \cdots + 1) K^m d(x_1, x_0) \leq \frac{K^m}{1-K} d(x_1, x_0)$, where $x_0 \in X$ and $x_{m+1} = f(x_m)$.

(b) Show that there exists exactly one $\bar{x} \in X$ such that $f(\bar{x}) = \bar{x}$. (Hint: Is $\{x_m\}$ a Cauchy sequence?)

(10) (a) Show the existence of a bijective map $\psi$ from $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}^+$, where $\mathbb{Z}$ is the set of integers by an explicit construction or picture of one.

(b) Consider the sequence $\{C_{(n,m)} : n, m \in \mathbb{Z}\}$ indexed by $(n,m) \in \mathbb{Z} \times \mathbb{Z}$. Let $\psi$ be your map in (a). Suppose that $\sum_{\psi(n,m)} C_{(n,m)}$ converges to some number in $\mathbb{R}$, but also assume that the series is NOT absolutely convergent $\left[\sum_{\psi(n,m)} C_{(n,m)} \text{ means it is the series } \sum_{n=1}^{\infty} t_n, \text{ where } t_n = C_{\psi^{-1}(n)}\right]$. Let $b \in \mathbb{R}$.

Show that there exists a bijective map $\psi_b : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ such that $\sum_{\psi_b(n,m)} C_{(n,m)} = b$.

(c) Conclude directly from (b) that the set of bijective maps from $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ is uncountable.