FIRST YEAR EXAM IN ANALYSIS

Jan. 1998

Do 10 of the following 11 problems.
Be sure to put each problem on a separate page.
Print your name on each page handed in.
Give complete justifications. Do not use arrows
or other unnecessary signs; use words for
explanations. Write complete sentences.
All work must be done in a neat and logical
fashion in order to obtain full credit.

① Let (x_n) be a sequence of real numbers and
suppose that $x_0 = \lim x_n$ exists (finite or $-\infty$)
and that $x_n > x_0$ for every n. Show that one
can change the order of the terms in the
sequence (x_n) to obtain a decreasing (i.e.
non increasing) sequence (y_n).
Does $\lim y_n$ exist? If yes, what is this
limit?
State the definition of $\lim x_n = +\infty$.

② Let $f: \mathbb{R} \to \mathbb{R}$ be a function and $x_0 \in [-\infty, +\infty]$.
Assume that for every decreasing (i.e. non
increasing) sequence (x_n) of rational
numbers with $x_n > x_0$ and $x_n \downarrow x_0$,
$\lim f(x_n)$ exists (finite or infinite).

(5) Let (f_n) be a sequence of real valued functions defined on the interval $[0,1]$ by

$$f_0(x) = \sqrt{x}\ , \quad f_{n+1}(x) = \sqrt{x + f_n(x)}\ .$$

Prove that the sequence (f_n) converges uniformly on $[0,1]$ and find its limit.

State the definition of uniform convergence of a sequence of functions.

Give a complete statement of the theorem used in the proof.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that if $G \subset \mathbb{R}$ is open, then $f(G)$ is also open.

Prove that f is strictly monotonic.

(7) Let $f : (0, +\infty) \to \mathbb{R}$ be a differentiable function. Suppose that

$$\lim_{x \to \infty} [\, f(x) + f'(x) \,] = A \in \overline{\mathbb{R}}\ .$$

Prove that $\lim_{x \to \infty} f(x) = A$ and $\lim_{x \to \infty} f'(x) = c$.

Hint: Use l'Hospital rule for $\dfrac{e^x f(x)}{e^x}$.

State l'Hospital theorem.