First year exam in Analysis
September 1997

Do each of the ten problems. Be sure to put each problem on a separate page. Print your name on each page handed in. Write compete sentences and use a complete and precise mathematical style. All work must be done in a neat and logical fashion in order to obtain full credit.

1. State and prove the monotone convergence theorem for nonnegative functions.

2. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$

   a) Find the points of continuity of $f$.
   b) Find the points where $f$ is differentiable.

3. Prove that there is a unique number $c > 1$ such that $\displaystyle\int_c^x \frac{dt}{\log t} = \log(\log x) + \sum_1^\infty \frac{(\log x)^n}{n(n!)}$, using the following steps:

   a) Prove that $\sum_1^\infty \frac{(\log x)^n}{n(n!)}$ converges for $x \geq 1$;
   b) Prove that $f(x) = \log(\log x) + \sum_1^\infty \frac{(\log x)^n}{n(n!)}$ is differentiable for $x > 1$ and that $f'(x) = \frac{1}{\log x}$.
   c) Prove that $f$ has a unique zero, $c > 1$.

4. Prove that a measurable function $f \geq 0$ is the limit of an increasing sequence of simple functions, and that a real, bounded, measurable function is the uniform limit of a sequence of simple funciton.

5. Let $f_n$ be a sequence of real function defined on the interval $[0, 3]$ by

$$f_n(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{n} \\ -x + \frac{2}{n} & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ x - \frac{2}{n} & \text{if } \frac{2}{n} \leq x \leq 3 \end{cases}$$

   and let $f : [0, 3] \to \mathbb{R}$ be defined by $f(x) = x$. Does $(f_n)$ converge pointwise to $f$ on $[0, 3]$? Does $(f_n)$ converge uniformly to $f$ on $[0, 3]$?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(t) \neq 1$ for every $t \in \mathbb{R}$. Prove that there is at most one point $x \in \mathbb{R}$ with $f(x) = x$. Give a complete statement of the theorems used in your proof.

7. Let $f : [0,1] \to \mathbb{R}$ be a monotone function. Show that for every $n$ we have

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leq \frac{1}{n}|f(1) - f(0)|.$$

State the theorem of approximation of the integral by Riemann sums.

8. Let $f : (a,b) \to \mathbb{R}$ be a monotone function. Prove that the set of points of discontinuity of $f$ is at most countable.

9. Let $E \subset \mathbb{R}$ be any set and $f : E \to \mathbb{R}$ a monotonic function. Prove that if the range $f(E)$ is an interval, then $f$ is continuous on $E$. Hint: prove separately left continuity and right continuity.

10. Give the definition of a compact set in a metric space.
Let $X$ be a compact metric space and $f : X \to \mathbb{R}$ a continuous function.
Show that $f$ attains its supremum and its infimum.