

DO EACH OF THE TEN PROBLEMS. Be sure to put each problem on a separate page. Print your name on every page handed in.

All work must be done in a neat and logical fashion in order to obtain credit. Write complete explanations and justify all steps by quoting theorems or supplying proofs. A gap in a proof is regarded as a serious mistake.

1. Let  $(S, \mathcal{M}, \mu)$  be a finite positive measurable space (so  $S \in \mathcal{M}$ ,  $\mu(S) < \infty$ , and for every  $E \in \mathcal{M}$ ,  $\mu(E) \geq 0$ ). The symmetric difference of two sets is  $A \Delta B = (A \cup B) \setminus (A \cap B)$ .

Let  $\{A_n\}_{n=0}^{\infty}$  be a sequence of sets in  $\mathcal{M}$  such that  $\lim_{n \rightarrow \infty} \mu(A_n \Delta A_0) = 0$ . Prove or disprove:  $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A_0)$ .

2. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3}.$$

(a) Prove that for any  $0 < R < \infty$ , the series above converges uniformly on  $[0, R]$ .

(b) For any  $x > 0$  and for integers  $n$  in the interval  $x/2 < n < 2x$ , show that

$$\frac{nx^2}{n^3 + x^3} > \frac{1}{18}.$$

What is  $\lim_{x \rightarrow \infty} f(x)$ ?

3. Let

$$f(x, y) = \begin{cases} (x^2 + y^2)^\alpha & \text{when } x \text{ and } y \text{ are both irrational} \\ 0 & \text{otherwise} \end{cases}$$

(a) With  $\alpha = 1$ , determine all values of  $(x, y)$  at which  $f$  is continuous.

(b) With  $\alpha = 1$ , determine all values of  $(x, y)$  at which  $f$  is differentiable.

(c) With  $\alpha = 1/2$ , determine all values of  $(x, y)$  at which  $f$  is continuous.

(d) With  $\alpha = 1/2$ , determine all values of  $(x, y)$  at which  $f$  is differentiable.

4. Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable function such that  $\lim_{x \rightarrow +\infty} f(x) = a$  and  $\lim_{x \rightarrow +\infty} f'(x) = b$ .

If  $-\infty < a < +\infty$ , what are the possible values for  $b$ ?

5. Let  $A \subset \mathbf{R}$  be any set, let  $I \subset \mathbf{R}$  be an interval, and let  $f: A \rightarrow I$  be a monotonically increasing function such that  $f(A) = I$ . Prove that  $f$  is continuous on  $A$ . (Hint: prove left and right continuity separately.)

6. This problem calls for examples of functions with certain properties. You should define your functions clearly and prove the properties you claim.

(a) Give an example of a function  $g: [0, 1] \rightarrow \mathbf{R}$  which is Riemann integrable but not continuous.

(b) Give an example of a function  $h: [0, 1] \rightarrow \mathbf{R}$  which is Lebesgue integrable but not Riemann integrable.

7. Is every closed subset of  $\mathbf{R}$  a countable union of compact sets? (Answer “yes” or “no,” and give a clear justification for your answer.)

8. Let  $S = \{f: [0, 1] \rightarrow \mathbf{R} : f \text{ is continuous and } \sup_{x \in [0, 1]} |f(x)| = 1\}$ . You may use the fact that

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

is a metric on the space of continuous real-valued functions on  $[0, 1]$ , and therefore  $d$  is a metric on  $S$ .

Show that  $S$  is not compact (with respect to this metric).

9. This problem concerns a trigonometric series.

(a) State the “partial summation formula” for a series of the form  $\sum a_n b_n$ .

(b) Prove the trigonometric identity

$$\sum_{n=1}^N \sin(nx) = \frac{\cos(x/2) - \cos(x \cdot (N + 1/2))}{2 \sin(x/2)},$$

for  $x \neq 2n\pi$ ,  $n \in \mathbf{Z}$ .

(c) Prove that

$$\sum_1^{\infty} \frac{\sin(nx)}{\sqrt{n}}$$

converges for all  $x \in \mathbf{R}$ .

(d) Does the series in (c) converge uniformly on  $[0, 2\pi]$ ? (Hint: Use Bessel’s inequality or the Riesz–Fischer theorem.)

10. Construct a compact set of real numbers whose limit points form a countable set.