First Year Analysis Examination

September 1996

Do each of the ten problems. Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit. Write complete explanations and justify all steps by quoting theorems or supplying proofs. A gap in a proof is regarded as a serious mistake.

1. Let X be an infinite set with metric defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. Is X a complete metric space? Prove your answer.

2. Let $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}, n = 1, 2, \ldots$. Prove that $s_n < 2$ for all n and that $\{s_n\}$ converges.

3. Consider the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Is it possible to form a rearrangement of this series, say $\sum\limits_{n=1}^{\infty} a_n$, such that $\sum\limits_{n=1}^{\infty} a_n = \frac{1}{2}$? Give reasons.

4. Let $f : [0,1] \to [0,1]$ be defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1 - x & \text{otherwise} \end{cases}$$

(a) Find the continuity points of f.
(b) Find the points where f is differentiable.

5. Let f be a real valued function defined on (a,b). Suppose that the derivative f' exists and is continuous on (a,b). Also Suppose that there exists a $w \in (a,b)$ with $|f'(w)| < 1$. Prove that there exists a neighborhood of $W, N_r(w)$ and a real number $k < 1$ such that for all x and y in $N_r(w)$, $|f(x) - f(y)| \leq k|x - y|$.

6. Let $\{f_n\}$ be a sequence of real valued functions defined for each positive integer n on $[0,3]$ as follows:

$$f_n(x) = x \text{ if } 0 \leq x \leq \frac{1}{n};$$
$$f_n(x) = -x + \frac{2}{n} \text{ if } \frac{1}{n} \leq x \leq x\frac{2}{n};$$
$$f_n(x) = x - \frac{2}{n} \text{ if } \frac{2}{n} \leq x \leq 3.$$

Define $f(x) = x$ on $[0,3]$. Does $\{f_n\}$ converge pointwise to f on $[0,3]$? Does $\{f_n\}$ converge unformly to f on $[0,3]$? Prove your answers.

7. Let $f : [-\pi, \pi) \to \mathbb{R}$ be defined by $f(x) = 1$ if $|x| < \frac{\pi}{2}$ and $f(x) = 0$ otherwise.

(a) Compute the Fourier series of f.

(b) Show $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

(c) Show $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

8. Let $\{a_1, a_2, \dots\}$ be an enumeration of the rationals in $[0,1]$. Let $I_j = (a_j - \frac{1}{2^{j+2}},\ a_j + \frac{1}{2^{j+2}}) \cap [0,1]$. Let $E = \bigcup_{j=1}^{\infty} I_j$. Show $m(E) \le \frac{1}{2}$. Conclude $[0,1] \sim E$ is a compact, uncountable set which contains no rationals.

9. Fix $0 < \alpha \le 1$. Let $E_0 = [0,1]$. Remove the middle $\frac{\alpha}{3}$ from E_0 and let E_1 be the union $[0, \frac{1}{2} - \frac{\alpha}{2\cdot 3}] \cup [\frac{1}{2} + \frac{\alpha}{2\cdot 3}, 1]$. Remove the middle $\frac{\alpha}{9}$ from these intervals, and let E_2 be the union $[0, \frac{\frac{1}{2}-\frac{\alpha}{2\cdot 3}}{2} - \frac{\alpha}{2\cdot 9}] \cup [\frac{\frac{1}{2}-\frac{\alpha}{2\cdot 3}}{2} + \frac{\alpha}{2\cdot 9}, \frac{1}{2} - \frac{\alpha}{2\cdot 3}] \cup [\frac{1}{2} + \frac{\alpha}{2\cdot 3}, \frac{\frac{1}{2}+\frac{\alpha}{2\cdot 3}+1}{2} - \frac{\alpha}{2\cdot 9}] \cup [\frac{\frac{1}{2}+\frac{\alpha}{2\cdot 3}+1}{2} + \frac{\alpha}{2\cdot 9}, 1]$. Continuing in this way obtain a sequence of sets $\{E_j\}_{j=0}^{\infty}$. Let $E = \bigcap_{j=0}^{\infty} E_j$. Find the Lebesgue measure of E.

10. Let X be a set and suppose μ is a finitely additive positive set function defined on a σ-ring $\mathcal{R}$ of subsets of X. Suppose μ has the following property:

> whenever $\{A_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of sets from $\mathcal{R}$ such that $\bigcap_{n=1}^{\infty} A_n = \phi$, then $\lim_n \mu(A_n) = 0$.

Show that μ is countably additive on $\mathcal{R}$.