First Year Analysis Examination

August 1995

Do each of the ten problems. Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. Let K be a compact subset of a metric space X. Prove that K is closed.

2. Let Y be the metric space consisting of all continuous, real valued functions defined on $[0,1]$ with metric
$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$
Let p be a fixed real number. Let E be the subset of Y which consists of all $f \in Y$ such that p is not in the range of f. Prove that E is an open subset of Y.

3. Let $E \subset R$ be a segment (open interval), and suppose $f : E \to R$ is monotonically increasing. Suppose also that $f(E)$ (the range of f) is a segment. Prove that f is continuous on E.

4. Suppose that $\{a_k\}$ is a sequence of non-zero real numbers, and
$$q = \lim_{k\to\infty} \frac{\log(\frac{1}{|a_k|})}{\log k} \text{ exists}$$
Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely if $q > 1$.
Hint: There is a real number p with $1 < p < q$.

5. Let $g : R \to R$ be defined by
$$g(x) = \begin{cases} x & \text{if } x < 0. \\ x+1 & \text{if } x \geq 0. \end{cases}$$
Does there exist a differentiable function $f : R \to R$ such that $f'(x) = g(x)$ for each $x \in R$?
Give complete verification of your answer.

6. Let $f : [0,1] \to [0,\infty]$ be continuous. If $\int_0^1 f(x)dx = 0$, what can you say about f?
Give complete verification of your answer.

7. If

$$I(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b) and if $\sum_{n=1}^{\infty} |c_n|$ converges, what can you say about the convergence of the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \quad a \leq x \leq b?$$

If $x \in [a, b]$ and $x \neq x_n$ for each n, is f continuous at x? Give complete verification of your answers.

8. Let X be a measurable space, and let f be a real valued function defined on X.
8a. Complete the following definition. The function f is said to be measurable if......
8b. Prove that f is measurable if and only if for every open subset V of R, $f^{-1}(V)$ is measurable.
Hint: Any open subset of R is a countable union of segments (open intervals).

9. Let (X, m, μ) be a measurable space, and let $f \geq 0$ be an integrable function (with respect to μ). Suppose $\int_X f \, d_\mu = 0$.
Prove that $f = 0$ almost everywhere (with respect to μ).

10a. State and prove Fatou's theorem.

10b. Show that the inequality appearing in the theorem may be strict.