FIRST YEAR EXAM IN ANALYSIS JANUARY 1995

DO EACH OF THE TEN PROBLEMS. Be sure to put each problem on a separate page. Print your name on every page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. If $\{f_n\}$ and $\{g_n\}$ are sequences of real-valued functions on a set E which converge uniformly to bounded functions f and g, respectively, show that $\{f_n g_n\}$ converges uniformly on E.

2. Let $A \subseteq \mathbf{R}$ be a non-empty set which has the property that every sequence in A has a subsequence that converges to a point of A. Show that A is bounded above and that the least upper bound of A is an element of A.

3. Let m denote Lebesgue measure on the real line and let $f: \mathbf{R} \to \mathbf{R}$ be a Lebesgue integrable function. Suppose that $\int_E f\, dm = 0$ for every measurable $E \subseteq \mathbf{R}$. Show that $f = 0$ almost everywhere.

4(a) Suppose that $\sum a_n$ is a series of non-negative real numbers whose partial sums are bounded. Prove that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent.

(b) Does the conclusion of (a) remain true if the non-negativity hypothesis on the a_n's is removed? Justify your answer.

5. Suppose that $p > 0$. Show that

$$\lim_{n\to\infty} \sqrt[n]{p} = 1.$$

6. Suppose that $\{f_n\}$ is a sequence of continuous functions $f_n: [0,1] \to \mathbf{R}$ which converges uniformly to $f: [0,1] \to \mathbf{R}$. Prove, starting from the definition of uniform convergence, that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx.$$

7. Let φ be a finitely additive set function defined on a σ-ring $\mathcal{R}$. Suppose that for any sequence of sets A_n from $\mathcal{R}$ such that $A_n \supseteq A_{n+1}$ for $n = 1, 2, 3, \ldots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, we have $\lim_{n\to\infty} \varphi(A_n) = 0$. Prove that φ is countably additive on $\mathcal{R}$.

8(a) State the Generalized Mean Value Theorem.
(b) Let $f: [0,1] \to \mathbf{R}$ be a continuous function which is differentiable at every point of $(0, 1)$.
Show that for each $n = 1, 2, 3, \ldots$ there exists $c \in (0,1)$ such that

$$f(1) - f(0) = \frac{f'(c)}{nc^{n-1}}.$$

9. Let $h: \mathbf{R} \to \mathbf{R}$ be defined by

$$h(x) = \begin{cases} 0, & x < -1 \\ 1, & -1 \leq x \leq 1 \\ 0, & 1 < x \end{cases}$$

Is there a differentiable function $f: \mathbf{R} \to \mathbf{R}$ such that $f'(x) = h(x)$?

10. Let $f: [0, \infty) \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} (1 + n^3 x)^{-1}, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Is f continuous at $x = 0$?
(b) Determine the largest interval on which f is continuous.
[Explain your answers to both questions carefully.]