First Year Exam in Analysis August 1994

Do each of the ten problems. Be sure to put each problem on a separate page. Print your name on every page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. Let $\{f_n\}$ be a sequence of Lebesgue integrable functions defined on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} |f_n|\, dm \leq \frac{1}{n^2} \qquad (n = 1, 2, \ldots)$$

Prove that $\sum_{n=1}^{\infty} f_n(x)$ converges for almost every $x \in \mathbf{R}$.

2. Define

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

for z complex. Prove that $E(z + w) = E(z)E(w)$ for all $z, w \in \mathbf{C}$. Justify all the steps in your argument.

3. A subset A of $\mathbf{R}^n$ is *convex* if whenever $x, y \in A$ and $0 < \lambda < 1$ then $\lambda x + (1 - \lambda)y \in A$. Prove that every convex subset of $\mathbf{R}^n$ is connected.

4. Let A and B be Lebesgue measurable subsets of $[0, 1]$. Suppose that the Lebesgue measures of A and B are

$$m(A) = \frac{1}{2},$$
$$m(B) = \frac{3}{4}.$$

What are the maximum and minimum possible values of $m(A \cap B)$?
Are all values between these extrema attained by $m(A \cap B)$?

5. Suppose that $a_n \geq 0$ for each n and that a_n is a decreasing sequence. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\sum_{n=1}^{\infty} 3^n a_{3^n}$$

converges.

6. Suppose that $\phi\colon [a,b] \to \mathbf{R}$ is Riemann integrable and positive, and that $f\colon [a,b] \to \mathbf{R}$ is continuous. Prove that there is a point $x \in [a,b]$ such that

$$\frac{\int_a^b f(t)\phi(t)\,dt}{\int_a^b \phi(t)\,dt} = f(x).$$

7. Suppose that $a < c < b$ and that $f\colon (a,b) \to \mathbf{R}$ is a twice continuously differentiable function.

(a) Compute

$$\lim_{t\to 0} \frac{f(c+t) - f(c-t)}{2t}.$$

(b) Compute

$$\lim_{t\to 0} \frac{f(c+2t) - 2f(c+t) + f(c)}{t^2}.$$

8. Suppose that $a < b$ and that $f\colon [a,b] \to \mathbf{R}$ is continuous. Prove that the image of $[a,b]$ under f is a closed and bounded interval.

9. Suppose that $\{f_n(x)\}$ is a sequence of continuous functions and that $\{f_n(x)\}$ converges uniformly to $f(x)$ on $[0,1]$. Prove that $f(x)$ is continuous on $[0,1]$.

10. Let

$$f_n(x) = \frac{1}{nx+1},$$

for $0 < x < 1$ and $n = 1, 2, 3, \ldots$

(a) Show that $f_n(x) \to 0$ monotonically in $(0,1)$.

(b) Is the convergence uniform?