FIRST YEAR EXAM IN ANALYSIS MAY 1994

DO EACH OF THE TEN PROBLEMS. Be sure to put each problem on a separate page. Print your name on every page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. Let X be a metric space such that for each countable set $A \subseteq X$, the closure $\overline{A}$ is compact. Prove that X is compact.

2a. Let f and g be continuous real-valued functions on a metric space X and let

$$E = \{x \in X : f(x) = g(x)\}.$$

Show that if E is dense in X then $f = g$ (i.e. $E = X$).

b. Does the conclusion of (a) remain true if f and g are continuous functions from X to another metric space Y, not necessarily $\mathbf{R}$?

3. Let $I = [0,1]$.

a. Prove that if $f: I \to I$ is continuous then there is at least one point $x \in I$ such that $f(x) = x$.

b. Is the same conclusion true if $[0,1]$ is replaced by the open interval $(0,1)$?

4. Suppose that $f: [a,b] \to \mathbf{R}$ is a continuous differentiable function such that $f(a) = f(b) = 0$ and $\int_a^b [f(x)]^2\,dx = 1$. Prove that:

a. $\int_a^b x f(x) f'(x)\,dx = -1/2$

b. $\left(\int_a^b [f'(x)]^2\,dx\right)\left(\int_a^b x^2 [f(x)]^2\,dx\right) > 1/4$

5. Let $\{a_n\}$ be a sequence of positive real numbers such that $\sum_1^\infty a_n = \infty$. Prove that

$$\sum_1^\infty \frac{a_k}{1 + a_k}$$

diverges, or give a counterexample.

6. Prove that if f is continuous on $\mathbf{R}$ then for any real number c the set $\{x \in X : f(x) \le c\}$ is closed and the set $\{x : f(x) < c\}$ is open.

7a. Find the value of $\sum_{n=1}^{\infty} x^n$ for $|x| < 1$.

b. Prove that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

8. If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.

9. Suppose that $\{f_n\}$ is a sequence of Lebesgue integrable functions defined on the real line $\mathbf{R}$ which converges uniformly to 0 on $\mathbf{R}$. Prove or disprove

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n(x)\, dm = 0,$$

where m denotes Lebesgue measure.

10. Let $(X, \mathcal{M}, \mu)$ be a measurable space. Suppose $\{f_n\}$ is a sequence of functions which are integrable and satisfy $\int_X |f_n|\, d\mu < 1/n^2$ for each n.

a. Let $g_k(x) = \sum_{i=1}^{k} |f_i(x)|$, for $x \in X$. Let $g(x) = \lim_{k\to\infty} g_k(x)$. Prove that g is integrable.

b. Prove that g is finite almost everywhere with respect to μ.

c. What can you conclude about the convergence of the series $\sum_{i=1}^{\infty} f_i$ on X?