FIRST YEAR EXAM IN ANALYSIS MAY 1993

DO AT MOST EIGHT OF THE TEN PROBLEMS. Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. Fill in the blank with one of the three words "compact," "bounded," "closed." *If $\{A_\alpha\}$ is a collection of subsets of the reals such that the intersection of every finite subcollection of $\{A_\alpha\}$ is nonempty, then $\cap_\alpha A_\alpha$ is nonempty.*

Give examples showing that neither of the choices you omitted would make the statement correct.

2. Let D be a bounded subset of the reals. If $f : D \to \mathbb{R}$ is continuous, must f be uniformly continuous?

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers with $a_n > 0$ and $b_n \geq 0$. If both the the sequence $\{\frac{b_n}{a_n}\}$ and the series $\sum a_n$ converge, does the series $\sum b_n$ converge?

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) + \dfrac{x}{2} & \text{if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

(1) Find $f'(0)$;
(2) find $f'(\frac{1}{2n\pi})$;
(3) Is there an interval containing 0 on which f is increasing?

5. Let $\{f_n\}$ be a sequence of Riemann integrable functions on the interval $[0, 1]$. Let

$$F_n(x) = \int_0^x f_n(t)dt \quad + \quad f_n(0).$$

Prove

(1) If $\{f_n\}$ is uniformly bounded (i.e. $|f_n(x)| \leq M$ for all $x \in [0, 1]$ and all $n \in \mathbb{N}$), then there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[0, 1]$.
(2) If each f_n is bounded, but not necessarily uniformly bounded, then there exists a subsequence $\{F_{n_k}\}$ which converges at each rational number q in $[0, 1]$.

6. Let $f_n(x) = \sin(nx)$, $0 \leq x \leq 2\pi, n = 1, 2, 3, \ldots$.

(1) Find $\int_0^{2\pi} |f_n - f_m|^2 dx$.
(2) Does there exist a subsequence $\{n_k\}$ such that $\{\sin(n_k x)\}$ converges for every $x \in [0, 2\pi]$.

7. Find a closed subset of the reals with positive Lebesgue measure that does not intersect the rationals.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Let m denote Lebesgue measure. Prove, for every $\epsilon > 0$ there exists a Lebesgue measurable set A_ϵ, such that $m(A_\epsilon) < \infty$ and $\int_B |f| dm < \epsilon$, where $B = \mathbb{R} \setminus A_\epsilon$.

9. Let $\{f_n\}$ be a sequence of measurable functions. Prove the set of points x at which $\{f_n(x)\}$ converges is a measurable set.

10. Show, if $f : [0,1] \to \mathbb{R}$ is Riemann integrable and $f(q) = 0$ for every rational number $q \in [0,1]$, then $\int_0^1 f dx = 0$. Is the hypothesis f is Riemann integrable needed? What happens if Riemann integrable is replaced by Lebesgue integrable?