First Year Exam in Analysis January 1993

Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit. Each of the 10 problems is worth 10 points.

1. Let X be a metric space. Suppose $\{E_n\}$ is a sequence of closed nonempty subsets of X such that $E_n \supset E_{n+1}$ for all n. Must $\bigcap_n E_n$ be nonempty?

2. Let X be a metric space.

(1) Give the definition for a sequence $\{x_n\}_1^\infty$ from X to be a Cauchy sequence.

(2) Let A be a subset of X such that given any $\epsilon > 0$, A can be covered by finitely many open balls in X of radius ϵ. Prove that every sequence of elements in A has a Cauchy subsequence.

3. Suppose $\sum_{n=1}^\infty a_n$ converges, but does not converge absolutely. Show, given a real number A, there exists a rearrangement $\sum_{n=1}^\infty a'_n$ of the series $\sum_{n=1}^\infty a_n$ such that $\sum_{n=1}^\infty a'_n$ converges to A.

4. Let I denote the interval $[0,1]$. Suppose $f : I \to I$ is continuous. Prove that $f(x) = x$ for at least one $x \in I$. Is the same conclusion true if I is the open interval $(0,1)$?

5. Let f be a continuously differentiable function on the interval $[0,1]$.

(1) Suppose $0 \le q < p \le 1$ and let $m = \min\{|f'(t)| : p \le t \le q\}$. Show

$$m(p-q) \le |f(p) - f(q)| \le \int_q^p |f'(t)|dt.$$

(2) For a partition $\mathcal{P} = \{0 = p_0 < p_1 < \cdots < p_n = 1\}$ of $[0,1]$, define

$$V(\mathcal{P}, f) = \sum_{j=1}^n |f(p_j) - f(p_{j-1})|.$$

Show,

$$L(\mathcal{P}, |f'|) \le V(\mathcal{P}, f) \le \int_0^1 |f'(t)|dt,$$

where $L(\mathcal{P}, |f'|)$ denotes lower sum.

(3) Let $V(f) = \sup\{V(\mathcal{P}, f) : \mathcal{P}\}$, where the supremum is taken over all partitions $\mathcal{P}$ of $[0,1]$. What is the relation between $V(f)$ and $\int_0^1 |f'|$?

6. For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Show that

$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx,$$

where $[x]$ denotes the greatest integer $\leq x$.

7. Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be sequences of real valued functions on E which converge uniformly to functions f_0 and g_0 respectively.

(1) Show, if, for each $j \geq 1$, there exists a constant a_j such that $|f_j(x)| \leq a_j$ for all $x \in E$, then there exits a constant A such that $|f_j(x)| \leq A$ for all $x \in E$ and $j \geq 0$.

(2) Does the sequence $\{g_n f_n\}_{n=1}^{\infty}$ converge uniformly to $f_0 g_0$?

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f = 0$ almost everywhere with respect to Lebesgue measure. What can you say about f? (Prove your answer).

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Let m denote Lebesgue measure. Prove that for every $\epsilon > 0$, there exists a Lebesgue measurable set A_ϵ, such that $m(A_\epsilon) < \infty$ and $|\int_B f dm| < \epsilon$ for every Lebesgue measurable set $B \subset \mathbb{R} \setminus A_\epsilon$.

10. Let (X, Σ, μ) be a measurable space. If $f \in \mathcal{L}^2(\mu)$, is $f \in \mathcal{L}(\mu)$? Does your answer change if $\mu(X) < \infty$?