FIRST YEAR EXAM IN ANALYSIS SEPTEMBER 1992

Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit. Each of the 10 problems is worth 10 points.

1. Let X be a metric space.

(a) Give the definition for a subset E of X to be compact.

(b) Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of points in X. Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x_0. Let $E = \{x_n : n = 0, 1, 2, ...\}$. Use the definition in part (a) to prove that E is compact.

2. Let X be a metric space.

(a) Give the definition for a sequence $\{x_n\}_1^{\infty}$ from X to be a Cauchy sequence.

(b) Let A be a subset of X such that given any $\epsilon > 0$, A can be covered by finitely many open balls in X of radius ϵ. Prove that every sequence of elements in A has a Cauchy subsequence.

3. Let D be a bounded subset of the real numbers. We do not assume that D is open or closed. Let f be a bounded continuous function defined on D. Is f uniformly continuous on D?

4. Given a series $\sum a_n$, define

$$p_n = \frac{|a_n| + a_n}{2} \quad \text{and} \quad q_n = \frac{|a_n| - a_n}{2}.$$

Show the series $\sum a_n$ converges absolutely if and only if both series $\sum p_n$ and $\sum q_n$ converge.

5. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers with $a_n > 0$ and $b_n \geq 0$ for all n. If both the sequence $\{\frac{b_n}{a_n}\}$ and the series $\sum a_n$ converge, does the series $\sum b_n$ converge?

6. Let $f_n(x)$ be a sequence of functions defined on a countable set $E = \{x_1, x_2, ..., x_n, ...\}$, which satisfies

$$|f_n(x_i)| \leq i, \quad \text{for all} \quad n = 1, 2, ... \text{ and } \quad i = 1, 2, ... \quad .$$

Find a subsequence of $\{f_n\}$ which converges for every x in E.

7. Let α be a monotonically increasing function on $[a, b]$. Suppose $x_0 \in (a, b)$ and α is continuous at x_0. Define f on $[a, b]$ by $f(x_0) = 1$ and $f(x) = 0$ if $x \neq x_0$. Is f integrable in the Riemann–Stieljes sense with respect to α? If so, find the value of this integral, $\int_a^b f d\alpha$.

8. Let Σ be a σ–ring of subsets of a set X. Let $\mu : \Sigma \to [0, \infty]$ be finitely additive. Suppose that if $A_n \in \Sigma$, $A_{n+1} \subset A_n$ and $\cap_1^\infty A_n$ is empty, then the sequence $\{\mu(A_n)\}_1^\infty$ converges to 0. Prove that μ is countably additive.

9. Suppose

(1) $|f(x, y)| \leq 1$ if $0 \leq x \leq 1$, $0 \leq y \leq 1$,
(2) for fixed x, $f(x, y)$ is a continuous function of y,
(3) for fixed y, $f(x, y)$ is a continuous function of x.

Put

$$g(x) = \int_0^1 f(x, y) dy \qquad (0 \leq x \leq 1).$$

Is g continuous?

10. Let A be Lebesgue measurable subset of the reals; and suppose $\mu(A)$ is finite, where μ is Lebesgue measure. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \mu\left((-\infty, x) \cap A\right).$$

(a) Is f continuous on $\mathbb{R}$?
(b) Is f uniformly continuous on $\mathbb{R}$?