FIRST YEAR EXAM IN ANALYSIS MAY 1992

Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit. Each of the 10 problems is worth 10 points.

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1. In a metric space X, let set $E \subset X$. Show $E' = (\overline{E})'$.

2. Let X be a metric space. Let $E \subset X$.

   a. Give the definition for E to be compact.

   b. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of points in X. Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x_0. Let $E = \{x_n : n = 0, 1, 2, ...\}$. Use definition in part (a) to prove that E is compact.

3.

   a. Suppose f is a uniformly continuous real valued function defined on $(0, 1]$. Show $\{f(1/n)\}_{n=1}^{\infty}$ converges.

   b. Define g on $(0, 1]$ by $g(x) = \sin\left(\frac{\pi}{2x}\right)$. Show g is not uniformly continuous.

4. Suppose that $\{g_n\}$ is a sequence of continuous functions on $[0, 1]$. Suppose that $g_n(x) \geq g_{n+1}(x)$, for each $x \in [0, 1]$ and each n. Assume that $\{g_n(x)\}$ converges to x^2 for each $x \in [0, 1]$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence from $[0, 1]$ and suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x_0. Prove that the sequence $\{g_n(x_n)\}_{n=1}^{\infty}$ converges to x_0^2.

5. Suppose $\{a_n\}$ is a sequence of positive real numbers such that Σa_n converges. Show that if k is a positive integer, then Σa_n^k converges.

6. Find all closed intervals on which the sequence $\{f_n\}$ converges uniformly, where

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}, x \in \mathbf{R}.$$

7. Let f be a real valued differentiable function on the interval (0, 1).

(a) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence from (0, 1) which converges to $x_0 \in (0,1)$. show if there exists a positive number c such that

$$f'(x_n) > f'(x_0) + c$$

for each n, then there exists a sequence $\{y_n\}$ which converges to x_0 such that $f'(y_n) = f'(x_0) + c$ for each n.

(b) Show that f' is continuous if and only if $(f')^{-1}(\{z\})$ is closed for each $z \in \mathbf{R}$.

8. Give an example of a sequence of Lebesgue integrable functions $\{f_n\}_{n=0}^{\infty}$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f_0 on $\mathbf{R}$, but $\{\int_{\mathbf{R}} f_n dx\}_{n=1}^{\infty}$ does not converge to $\int_{\mathbf{R}} f_0 dx$.

9. Let f be a bounded real valued Lebesgue measurable function defined on $\mathbf{R}$. let $\epsilon > 0$ be given. Prove that there exists a simple Lebesgue measurable function g such that $|f(x) - g(x)| < \epsilon$, for all $x \in \mathbf{R}$.

10. Establish the following version of the Dominated Convergence Theorem. Let (X, Σ, μ) be a measurable space and $\{g_n\}_{n=1}^{\infty}$ a sequence of integrable functions on X (belong to $\mathcal{L}(\mu)$) such that $\{g_n(x)\}_{n=1}^{\infty}$ converges to the integrable function $g(x)$ for each $x \in X$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $|f_n(x)| \leq g_n(x)$, for each $x \in X$ and n. Suppose $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ for each $x \in X$. Show, if

$$\int_X g_n d\mu \longrightarrow \int_X g d\mu,$$

then

$$\int_X f_n d\mu \longrightarrow \int_X f d\mu.$$

(Suggestion: Modify the standard proof of Dominated Convergence using Fatous' Lemma).