

First Year Exam - Analysis  
Jan. 1992

Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be written in a neat and logical fashion to obtain credit

1. Let  $\{K_\alpha\}$  be a collection of compact subsets of a metric space, such that the intersection of every finite subcollection of  $\{K_\alpha\}$  is nonempty. Prove that  $\bigcap K_\alpha$  is nonempty.
2. Let  $\sum a_n$  be a series of real numbers which converges, but not absolutely. Construct a rearrangement  $\sum a'_n$  which converges to the number 7.
3. Suppose  $a_n > 0$ ,  $n = 1, 2, \dots$ , and assume that  $\sum a_n$  diverges. Prove  $\sum (a_n/1 + a_n)$  diverges.
4. Suppose that  $\{f_n\}$  is a sequence of continuous functions which converges to the continuous function  $f$  at each  $x \in [0, 1]$ . Suppose  $f_n(x) \geq f_{n+1}(x)$  for each  $x \in [0, 1]$ ,  $n = 1, 2, \dots$ .  
Suppose that the sequence  $\{x_n\}$  of numbers from  $[0, 1]$  converges to  $x_0 \in [0, 1]$ .  
Prove that  $\lim_{n \rightarrow \infty} f_n(x_n) = f(x_0)$ .
5. Recall the definition of the unit step function,

$$I(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Suppose  $c_n \geq 0$  and  $\sum_{n=1}^{\infty} c_n$  converges and suppose that  $\{s_n\}$  is a sequence from the bounded interval  $[a, b]$ . Let  $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$ .

Let  $f$  be continuous at each  $s_n$ , where  $f : [a, b] \rightarrow \mathbf{R}$  is bounded.  
Is  $f$  Riemann-Stieltjes integrable on  $[a, b]$  with respect to  $\alpha$ ?

6. Call a mapping  $f$  of  $X$  into  $Y$  open if  $f(V)$  is an open set in  $Y$  whenever  $V$  is an open set in  $X$ .

Prove that every continuous open mapping of  $\mathbf{R}$  into  $\mathbf{R}$  is monotonic.

7. Use the identity  $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ ,  $x \in (-1, 1)$  to prove that  $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)3^{\frac{2n+1}{2}}} = \frac{\pi}{6}$ .

8. Let  $\Sigma$  be a  $\sigma$ -ring of subsets of a set  $X$ . Let  $\mu : \Sigma \rightarrow [0, \infty]$  be finitely additive. Suppose that if  $A_n \in \Sigma$ ,  $A_{n+1} \subset A_n$  and  $\bigcap A_n = \phi$ , then  $\lim_n \mu(A_n) = 0$ . Prove that  $\mu$  is countably additive.

9. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be a Lebesgue-integrable function. Let  $m$  denote Lebesgue measure.

Prove that for every  $\varepsilon > 0$ , there exists a set  $A_\varepsilon$  which is Lebesgue measurable,  $m(A_\varepsilon) < \infty$  and  $|\int_B f dm| < \varepsilon$  whenever  $B \subset \mathbf{R} - A_\varepsilon$ , and  $B$  is a measurable set.

10. Let  $B$  denote the Borel subsets of  $(0, 1)$ , and let  $m$  be Lebesgue measure on  $B$ . Let  $\mu$  be a measure on  $B$  such that

$$\mu((a, b)) = \frac{1}{2}(b - a), \text{ for every } 0 < a < b < 1.$$

Prove  $\mu = \frac{1}{2}m$  on  $B$ .