First Year Exam - Analysis
Sept. 1991

All work must be presented in a neat and logical manner. Put each problem on a separate sheet. Print Name on each sheet.

1. (a) Suppose $f : (0,1] \to \mathbf{R}$ is uniformly continuous.
   Show that $\{f(\frac{1}{n})\}_{n=1}^{\infty}$ converges.

   (b) Define $g : (0,1] \to \mathbf{R}$ by $g(x) = \sin(\frac{\pi}{2x})$. Show g is not uniformly continuous.

2. (a) Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n}, \text{ where } n > 0 \text{ and g.c.d. } (m,n) = 1. \end{cases}$$

   Show f is continuous at each irrational number.

   (b) Define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational.} \end{cases}$$

   Is $f + g$ Riemann integrable on $[0,1]$?

3. In a metric space X, let E' denote the set of all limit points of a set $E \subset X$.
   Show that E' is closed and that $E' = (\overline{E})'$.

4. Investigate the convergence or divergence of Σa_n if

   (a) $a_n = \sqrt{n+1} - \sqrt{n}$

   (b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$

   (c) $a_n = (\sqrt[n]{n} - 1)^n$.

5. Suppose $a_n > 0$ and Σa_n diverges. Prove that $\Sigma \dfrac{a_n}{1 + a_n}$ diverges.

6. Prove that $e^x < \dfrac{1}{1-x}$ whenever $x < 1$ and $x \neq 0$.

7. Prove that

$$ln\, 2 = \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}.$$

(Hint: Use the identity $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

8. Let f be a nonnegative measurable function defined on the measurable set E.
Suppose $\int_E f dm = 0$.
Prove that $f(x) = 0$ almost everywhere on E.

9. Let A be a Lebesgue measurable set. Let $\varepsilon > 0$ be given. Prove that there exists an open set G such that $G \supset A$ and $m(G - A) < \varepsilon$.

10. Give an example of a sequence of Lebesgue integrable functions $f_n, n = 0, 1, 2, \ldots$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f_0 on $\mathbb{R}$, but $\int_R f_n \not\to \int_R f_0$.