1st Year Analysis Examination
May 28, 1991

Do each problem on a separate sheet and write your name on each sheet. Also provide a cover page listing the problems you have attached. Each problem is worth 10 points, and you will be graded on a total of 100 points. (Justify all answers to receive credit.)

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1. Let $C_1, \dots, C_n$ be compact subsets of a metric space (X, d). Show that $\bigcup_{i=1}^{n} C_i$ is compact in (X, d).

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2. Suppose that $c_0 \geq c_1 \geq c_2 \geq \dots$ and $\lim_{n \to \infty} c_n = 0$. Show that $\Sigma c_n z^n$ converges at every point on the unit circle $|z| = 1$, except possibly at $z = 1$.

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3. a) Let $f : (0, 1] \to \mathbf{R}$ be uniformly continuous and let $\{x_n\}_{n \in \mathbf{N}} \subset (0, 1]$ converges to 0 in $[0, 1]$. Show that $\{f(x_n)\}_{n \in \mathbf{N}}$ converges.

   b) Is $\sin(\frac{1}{x})$ uniformly continuous on $(0, 1]$? Prove your answer is correct.

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4. Let $g : \mathbf{R} \to \mathbf{R}$ satisfy $|g'(x)| \leq M < +\infty$ for all $x \in \mathbf{R}$. Is it true that $f(x) \equiv x + \varepsilon g(x)$ is one–to–one for all sufficiently small $\varepsilon > 0$? Prove your answer.

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5. Let $f : [1, \infty) \to [0, \infty)$ be monotone decreasing.

   a) Is f Riemann integrable on $[a, b] \subset [1, \infty)$ for any $b > a$?
   b) Is f Lebesgue integrable on $[a, b] \subset [1, \infty)$ for any $b > a$?
   c) Does the sequence $\{\gamma_n\}_{n \in \mathbf{N}}$ with

$$\gamma_n \equiv \sum_{j=1}^{n} f(j) - \int_1^n f(t)dt$$

   converge?

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6. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and define $f_n(t) = f(nt)$, $n \in \mathbf{N}$. If $\{f_n\}_{n \in \mathbf{N}}$ is equicontinuous on $[0, 1]$, what conclusions can be drawn about f?

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7. Let $f(x)$ be a periodic function of period 2π with $f(x) = x^2$ for $|x| \le \pi$.

   a) Find the trigonometric Fourier series expansion of $f(x)$.
   b) At which points does this Fourier series converge to f?
   c) Find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (Justify your answers)

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8. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n : X \to \mathbf{R}$ be a measurable function for each $n \in \mathbf{N}$. Let $A \subset X$ be the set of points x for which $\{f_n(x)\}_{n\in\mathbf{N}}$ converges. Is A measurable? Justify your answer.

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9. Is it true that $f, g \in \mathcal{L}$ on $[a, b] \subset \mathbf{R}$ implies $f \cdot g \in \mathcal{L}$ on $[a, b]$? Justify your answer.

   [Note: $\mathcal{L}$ is the space of complex-valued functions integrable with respect to the usual Lebesgue measure on the reals.]

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10. Let $f : [a, b] \to \mathbf{R}$ have a continuous derivative f' and satisfy $f(a) = f(b) = 0$, as well as $\int_a^b f^2(x)dx = 1$.

    (a) Calculate $\int_a^b x f(x) f'(x) dx$.

    (b) Show that $\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx > \frac{1}{4}$.