1st Year Analysis Examination
January 23, 1991

Do each problem on a separate sheet and write your name on each sheet. Also provide a cover page listing the problems you have attached. Each problem is worth 10 points, and you will be graded on a total of 100 points. (Justify all answers to receive credit.)

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1. Give an example of:
   a) A closed set that is not perfect.
   b) A perfect set that is not closed.
   c) A countable set with a countable set of limit points.

2. Show that a metric space is disconnected if and only if it contains a set that is both open and closed.

3. For each natural number n, define the function

   $$f_n(x) = \frac{nx}{1+n^4x^4} \qquad \text{on the interval } [0,1] \subset \mathbf{R}.$$

   a) Does the sequence $\{f_n(x)\}_{n \epsilon \mathbf{N}}$ converge uniformly on $[0,1]$?

   b) Does the sequence $\{f_n(x)\}_{n \epsilon \mathbf{N}}$ converge uniformly on any subintervals of $[0,1]$? If so, what are they?

4. Given a continuous function $f : \mathbf{R} \to \mathbf{R}$ such that $\int_{-a}^{a} f(t)t^{2m}\,dt = 0$ for all $m \epsilon \mathbf{N}$, show that f is an odd function on the interval $[-a, a]$.

5. Let D be a compact, connected region in $\mathbf{R}^2$. Let f and g be continuous, bounded, real-valued functions on D with $g(x) \geq 0$ for all $x \epsilon D$. Show that there is a point $x_0 \epsilon D$ such that

   $$\iint_D fg\; d^2x = f(x_0) \iint_D g\; d^2x.$$

   Can the assumption $g(x) \geq 0$ be removed without changing the conclusion? If yes, prove it. If not,then give a counterexample.

6. Assume that $\sum_{n=1}^{\infty} \frac{a_n^2}{n}$ converges, where $\{a_n\}_{n \epsilon \mathbf{N}} \subset \mathbf{R}$. Does the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n$ exist? If so, what is it?

7. Let

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbf{Q} \text{ (the rationals)} \\ 0, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

a) Does $f'(0)$ exist?
b) Does $f'(x)$ exist for $x \neq 0$?
c) Find those x where f has discontinuities of the first kind, of the second kind.

8. Let [x] denote the largest integer less than or equal to x (the integer part of x), and let $\alpha(x) = x[x^2]$. Find $\int_0^2 x^2 d\alpha(x)$.

9. Let $C^1(\mathbf{R})$ be the set of all real-valued functions on $\mathbf{R}$ that are <u>continuously differentiable</u> and have compact support.
a) Show that $C^1(\mathbf{R})$ is not an empty set.
b) Is $C^1(\mathbf{R})$ dense in $\mathcal{L}^2(\mathbf{R}, dx)$? Prove it.

10. With f and g complex–valued functions in $\mathcal{L}^2(X, \mu)$, are the following two conditions equivalent?
(i) $|\int fg^* d\mu|^2 = (\int |f|^2 d\mu)(\int |g|^2 d\mu)$
(ii) There exists a constant λ such that $f = \lambda g$ almost everywhere.