1st Year Analysis Examination
May 21, 1990

Do each problem on a separate sheet and write your name on each sheet. Also provide a cover page listing the problems you have attached. Each problem is worth 10 points, and you will be graded on a total of 100 points.

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1. Suppose $f: [0,\infty) \to \mathbb{R}$ is continuous. Is it true that if $\lim_{x\to\infty} f(x) = 0$, then f is uniformly continuous.

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2. Find all values of z, in the complex plane, for which the series

$$\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 \, 4^n}$$

converges. Draw a picture of your result.

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3. Let X (resp. Y) be the space of all continuous (resp. continuously differentiable) real-valued functions on the closed interval $[0,1]$; and for every pair (f,g) of functions in X, let

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

(a) Is (X,d) a complete metric space? (explain)

(b) Is (Y,d) a complete metric space? (explain)

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4. Suppose $f:(0,1) \to \mathbb{R}$ is differentiable. Is it true that if f' is monotonic, then f' is continuous.

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5. Suppose that $f(x) \geq 0$ and that f decreases monotonically for $x \geq 1$. Show that the integral

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges.

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6. Let $\{f_n\}$ be an equicontinuous sequence of functions on $[0,1]$. Suppose that $\{f_n\}$ is pointwise convergent on $[0,1]$. Is $\{f_n\}$ uniformly convergent on $[0,1]$? Prove this fact, <u>or</u> give a counterexample.

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7. Let E be a Lebesgue measurable subset of $[0,1]$. Let m denote Lebesgue measure, and suppose $0 \leq \alpha \leq m(E)$. Prove that there exists a Lebesgue measurable set $F \subseteq E$ such that $M(F) = \alpha$. [<u>Hint:</u> Consider the function

$$f(x) = m(E \cap [0,x]),\quad 0 \leq x \leq 1]$$

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8. True or false: If f is a nonnegative function defined on R and

$$\int_{\mathbb{R}} f\,dx < \infty$$

then $\lim_{|x| \to \infty} f(x) = 0$?

Justify your answer.

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9. Let $f: [0,1] \to \mathbb{R}$ be a Lebesgue measurable function. Prove that there exists a Borel measurable function g such that $g = f$ a.e. on $[0,1]$ with repect to Lebesgue measure.

[Hint: Prove first when f is a simple function. For the general case, express f as the limit of simple functions].

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10. Let f be a nonnegative measurable function defined on R. Prove that if

$$g(x) = \sum_{n=-\infty}^{+\infty} f(x+n)$$

is integrable, then $f = 0$ a.e.