

First Year Algebra Exam**September 10, 2009**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G, H be finite groups with orders m, n .

- (a) Prove that if m and n are coprime then

$$\text{Aut}(G \times H) \cong \text{Aut}(G) \times \text{Aut}(H).$$

- (b) Give an example where m and n are not relatively prime and

$$\text{Aut}(G \times H) \not\cong \text{Aut}(G) \times \text{Aut}(H).$$

Justify your claims.

2. Say that the group G is supersolvable if there are subgroups

$$0 = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_{k-1} \leq H_k = G$$

of G such that $H_i \trianglelefteq G$ and H_i/H_{i-1} is cyclic for $1 \leq i \leq k$.

- (a) Give an example of a nonabelian supersolvable group. Justify your answer.
- (b) Prove that if G is a nontrivial finite supersolvable group then G contains a cyclic normal subgroup of prime order.
- (c) Prove that the alternating group A_4 is solvable, but not supersolvable.

3. Prove that if G is a nontrivial finite p -group then its center $Z(G)$ is nontrivial. Use this fact to prove that every finite p -group is nilpotent.
4. Let G be a group of order $385 = 5 \cdot 7 \cdot 11$. Prove that G contains a normal Sylow 11-subgroup, and that every Sylow 7-subgroup of G lies in the center of G .
5. Let R be a ring with 1 and let $I \subset R$ be a proper (2-sided) ideal on R . Prove that there is a maximal ideal M of R which contains I .

6. For each of the following polynomials $f(X) \in \mathbb{Q}[X]$, either prove that $f(X)$ is irreducible or give a nontrivial factorization of $f(X)$.
- (a) $f(X) = X^4 - 6X^2 + 15X - 21$
 - (b) $f(X) = X^4 - 6X^2 + 15X - 23$
 - (c) $f(X) = X^4 + X^3 + X^2 + X + 1$
 - (d) $f(X) = X^4 + 4$
7. Let R be a ring with 1, let F be an R -module, and let S be a subset of F such that every $y \in F$ can be written uniquely in the form $y = \sum_{x \in S} a_x x$ where a_x are elements of R all but finitely many of which are 0. Prove that for every R -module M and every function $\phi : S \rightarrow M$ there is a unique R -module homomorphism $\Phi : F \rightarrow M$ such that $\Phi|_S = \phi$.
8. Let F be a finite field with q elements and let V be a vector space over F such that $\dim_F(V) = n < \infty$. Let $0 \leq k \leq n$. Determine with proof the number of subspaces $W \subset V$ such that $\dim_F(W) = k$.
9. Give a representative for each conjugacy class in the group $\mathrm{GL}_3(\mathbb{F}_2)$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is the field with two elements.
10. Let M/L and L/K be field extensions of finite degree. Prove that

$$[M : K] = [M : L][L : K].$$