First Year Algebra Exam **May 12, 2009**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a cyclic group. Prove that every subgroup of G is cyclic. (Be sure to consider infinite cyclic groups as well as finite ones.)

2. Prove that the alternating group A_5 is simple. Prove that A_4 is solvable.

3. Let G be a group of order $132 = 2^2 \cdot 3 \cdot 11$. Prove that G has a nontrivial normal Sylow subgroup.

4. Show that there are four different homomorphisms $\phi : Z_2 \to \mathrm{Aut}(Z_8)$, and prove that the corresponding semidirect products $Z_8 \rtimes_\phi Z_2$ are pairwise nonisomorphic. (Hint: Consider the number of elements of order 2.)

5. Let R be an integral domain and let P be a prime ideal in R.

   (a) Prove that the set $D = R \setminus P$ is closed under multiplication.

   (b) Prove that the fraction ring $D^{-1}R$ has a unique maximal ideal.

6. Let R be an integral domain. State and prove Eisenstein's criterion for the irreducibility of a monic polynomial in $R[X]$.

7. Let R be an integral domain, let M be an R-module, and let $0 \leq n < \infty$. Say $\mathrm{rank}_R(M) = n$ if the following two conditions hold: (i) There exists a subset of M with cardinality n which is linearly independent over R. (ii) Every subset of M with cardinality $> n$ is linearly dependent over R.

   (a) Prove that if M is an R-module which is free on a set S with cardinality n then $\mathrm{rank}_R(M) = n$.

   (b) Give an example of an integral domain R and an R-module M such that $\mathrm{rank}_R(M) = 1$ but M is not a free R-module.

8. Let V be a vector space of the field F and let S be a set which spans V. Use Zorn's Lemma to prove that there is a basis for V which is contained in S.

9. Give a representative for each similarity class of 4×4 nilpotent matrices with entries in $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$. (We say that a square matrix A is nilpotent if $A^n = 0$ for some $n \geq 1$.)

10. Let E/F be a field extension and let $\alpha \in E$. Prove that if $[F(\alpha) : F]$ is an odd integer then $F(\alpha^2) = F(\alpha)$. Give an example where $[F(\alpha) : F]$ is even and $F(\alpha^2) \neq F(\alpha)$.