First Year Algebra Exam **January 20, 2009**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let p be a prime. Prove that every finite group has a Sylow p-subgroup.

2. Let $p > 3$ be a prime which is congruent to 3 modulo 4. Determine all groups of order $4p$ up to isomorphism.

3. (a) State the Fundamental Theorem of Finitely Generated Abelian Groups in the elementary divisor form.

   (b) Determine the number of isomorphism classes of abelian groups of order $2^4 \cdot 3^5 \cdot 7^6$. (Please don't write down the actual groups.)

4. Justify each of the following with appropriate subgroup series:

   (a) Give an example of a group which is nilpotent of class 3.

   (b) Give an example of a solvable group which is not nilpotent.

   (c) Give an example of a group which is neither simple nor solvable.

5. Let $R = (R, +, \cdot)$ be a ring. We define the opposite ring R^{op} to be $(R, +, *)$, where $x * y = y \cdot x$ for $x, y \in R$.

   (a) Prove that the identity map gives an isomorphism from R to R^{op} if and only if R is commutative.

   (b) Let $R = \mathrm{M}_2(\mathbb{Q})$. Give an isomorphism from R to R^{op}.

   (c) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Q} \right\}$. Prove that R is not isomorphic to R^{op}.

6. (a) Define what it means for a ring R to be a Euclidean Domain.

   (b) Let F be a field. Prove that $F[X]$ is a Euclidean Domain.

7. Let R be a ring with 1, let F be a free R-module of rank $n < \infty$, and let M be an R-module. Prove that there is an isomorphism of R-modules $\mathrm{Hom}_R(F, M) \cong M^n$.

8. Let V be a vector space over a field F and let S be a subset of V which spans V. Use Zorn's Lemma to prove that there is a subset of S which is a basis for V.

9. Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ denote the field with two elements. Give a representative for each conjugacy class in the group $\mathrm{GL}_4(\mathbb{F}_2)$. Justify your answer.

10. Let $\alpha \in \mathbb{C}$ be a root of $f(X) = X^3 - 3X^2 + 6 \in \mathbb{Q}[X]$. Express each of the following in the form $a + b\alpha + c\alpha^2$ with $a, b, c \in \mathbb{Q}$.

    (a) $(2 - \alpha)(1 + \alpha^2)$

    (b) α^{-1}

    (c) $(2 + \alpha)/(1 + \alpha^2)$