

**First Year Algebra Exam****September 2, 2008**

Instructions: Answer seven questions; please do not turn in more than seven. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (a) Give an example of a finite group  $G$  and a subgroup  $H \leq G$  such that  $H$  is not normal in  $G$  and the index  $|G : H|$  is prime.  
(b) Let  $G$  be a finite group of order  $n$  and let  $p$  be the smallest prime which divides  $n$ . Prove that if  $H$  is a subgroup of  $G$  of index  $p$  then  $H$  is normal in  $G$ .
2. Let  $G$  be a group and let  $H$  be a subgroup of  $G$ . Say that  $H$  is *fully invariant* if  $\phi(H) \leq H$  for every homomorphism  $\phi : G \rightarrow G$ .
  - (a) Prove that if  $H$  is a fully invariant subgroup of  $G$  then  $H$  is a characteristic subgroup of  $G$ .
  - (b) Prove that the commutator  $G' = [G, G]$  is a fully invariant subgroup of  $G$ .
  - (c) Let  $G = Z_2 \times S_3$ . Prove that  $Z(G)$  is not a fully invariant subgroup of  $G$ .
3. Let  $p, q, r$  be primes such that  $p < q < r$  and let  $G$  be a group of order  $pqr$ . Prove that  $G$  has a normal Sylow subgroup.
4. Let  $G$  be a cyclic group of finite order  $n \geq 1$ . Prove that  $\text{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ . Must  $\text{Aut}(G)$  be cyclic? Prove or give a counterexample.
5. Let  $p$  be a prime number.
  - (a) Compute the order of the general linear group  $GL_n(\mathbb{Z}/p\mathbb{Z})$ .
  - (b) Compute the order of the special linear group  $SL_n(\mathbb{Z}/p\mathbb{Z})$ . (This is the subgroup of  $GL_n(\mathbb{Z}/p\mathbb{Z})$  consisting of matrices which have determinant 1.)
6. Let  $R$  be a commutative ring with  $1 \neq 0$  and let  $I, J$  be ideals in  $R$  such that  $I + J = R$ . Prove the Chinese Remainder Theorem, i. e.,
$$(R/I) \times (R/J) \cong R/(I \cap J).$$
7. Prove that the ring of Gaussian integers  $\mathbb{Z}[i]$  is Euclidean.
8. Let  $F$  be a field and let  $g(X) \in F[X]$  be a nonzero polynomial. Prove that the  $F[X]$ -module  $F[X]/(g(X))$  is irreducible if and only if the polynomial  $g(X)$  is irreducible in  $F[X]$ .
9. Let  $A$  be an  $n \times n$  matrix with entries in a field such that  $A^k = 0$  for some  $k \geq 1$ . Prove that  $A^n = 0$ .
10. Let  $n \geq 1$ . Prove that there is at least one irreducible polynomial of degree  $n$  over  $\mathbb{Q}$ . Deduce that  $\mathbb{Q}$  has at least one extension of degree  $n$ .