First Year Algebra Exam **May 15, 2008**

Instructions: Answer seven questions; please do not turn in more than seven. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a group, let H, K be subgroups of G, and recall that HK is the set of products hk, where $h \in H$ and $k \in K$.
   (a) Prove that if H and K are finite then $|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$.
   (b) Prove that HK is a subgroup of G if and only if $HK = KH$.
2. (a) Show that every element of the alternating group A_5 has order 1, 2, 3, or 5.
   (b) Determine the conjugacy classes of A_5.
   (c) Prove that A_5 is simple.
3. Let $D_8 \cong \langle r, s : r^4 = s^2 = 1, rs = sr^{-1} \rangle$ be the dihedral group with 8 elements.
   (a) Determine the automorphism group $\mathrm{Aut}(D_8)$. Describe how the elements of $\mathrm{Aut}(D_8)$ act on the elements of D_8.
   (b) Prove that the inner automorphism group $\mathrm{Inn}(D_8)$ is not a characteristic subgroup of $\mathrm{Aut}(D_8)$.
4. Prove that every group of order 5000 is solvable.
5. Let R be a commutative ring with $1 \neq 0$. Prove that R contains a maximal ideal.
6. Let R be a UFD and let $f(X), g(X) \in R[X]$.
   (a) Define what it means for $f(X)$ to be primitive.
   (b) Prove that $f(X)g(X)$ is primitive if and only if $f(X)$ and $g(X)$ are both primitive.

7. Let R be a ring with 1, let M be an R-module, let F be a free R-module, and let $\phi : M \to F$ be a surjective module homomorphism. Prove that there is a submodule N of M such that $N \cong F$ and $M = \ker(\phi) \oplus N$.

8. Let F be a finite field with q elements, and let V be a vector space over F of dimension $n < \infty$.

   (a) For each k from 0 to n, determine with proof the number of subspaces of V of dimension k.

   (b) Determine with proof the number of invertible linear transformations from V to itself.

9. Find a representative for each conjugacy class of elements of order 4 in $\mathrm{GL}_5(\mathbb{Q})$.

10. Let $K \subset L \subset M$ be fields. Prove that $[M : K] = [M : L][L : K]$. In particular, show that $[M : K]$ is infinite if and only if either $[M : L]$ is infinite or $[L : K]$ is infinite.