

First Year Algebra Exam – January 2008

Time allowed: 240 minutes

Do **seven** of the following ten problems. Please do not turn in more than seven problems. You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, \mathbf{Q} is the set of all rational numbers.

1. State and prove Cayley's Theorem about finite groups.
2. Classify groups of order $2007 = 3^2 \cdot 223$ up to isomorphism.
3. Let G be a finite abelian 3-group. Assume that G has exactly **two** elements of order 3. Prove that G is cyclic. (**Hint** : You might want to use the structure theorem for finite abelian groups.)
 4. a) Define *Euclidean domain* and *principal ideal domain*.
b) Prove that any Euclidean domain is a principal ideal domain.
5. Let R be a ring. An R -module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M .
 - a) Prove Schur's Lemma: *Suppose the R -modules M and N are irreducible. Then every nonzero homomorphism $T : M \rightarrow N$ is an isomorphism.*
 - b) Let M be an irreducible R -module. Deduce from a) that $\text{End}_R(M)$, the set of all (module) homomorphisms $M \rightarrow M$, is a division ring.
6. Let A be a commutative ring with 1, and let I and J be ideals of A .
 - a) Define the product IJ and show that it is an ideal.
 - b) Prove that $IJ \subseteq I \cap J$, and give an example where equality does not hold.
7. How many conjugacy classes of 2-elements are there in the group $GL_5(\mathbf{F}_2)$ of 5 by 5 matrices over the field \mathbf{F}_2 of two elements? Justify your answer and give a representative for each conjugacy class. (Recall an element g of a group G is a *2-element* if $g^{2^n} = 1$ for some integer $n \geq 0$.)
8. Let V be a vector space of dimension 2008 over \mathbf{Q} . Does V have any linear transformation T such that $5T^{-1} = T^2 + 5T$? Justify your answer.
9. Find a splitting field \mathbf{F} for $x^4 - 7$ over \mathbf{Q} , and find $[\mathbf{F} : \mathbf{Q}]$.
10. Let \mathbf{F} be a field and \mathbf{E} be an extension of \mathbf{F} of degree 2009. Prove that $\mathbf{F}(u) = \mathbf{F}(u^3)$ for any $u \in \mathbf{E}$.