First Year Algebra Exam – Sept. 2007

Time allowed: 240 minutes

Do **seven** of the following ten problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Z}$, resp. $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{C}$, is the set of all integers, resp. positive integers, rational numbers, complex numbers.

1. State and prove Eisenstein's criterion for polynomials over $\mathbf{Z}$. (In the proof you may use Gauss' lemma).

2. Is there any simple group of order 105? Justify your answer.

3. Classify, up to isomorphim, the groups of order 55.

4. Let R be a ring with identity $1 \neq 0$ and let $I \neq R$ be an ideal in R. Prove that I is contained in a maximal ideal of R.

5. How many conjugacy classes of elements of order 4 are there in the group $GL_4(3)$ of 4 by 4 invertible matrices over $\mathbf{Z}/3\mathbf{Z}$? Justify your answer and give a representative for each conjugacy class.

6. Let I be the smallest ideal of the ring $S = \mathbf{Z}[\sqrt{-5}]$ that contains 3 and $1 + \sqrt{-5}$. Prove that I is not a principal ideal of S.

7. Let $p(x) = x^4 - 3x^2 + 6x + 12$. Prove that $R = \mathbf{Q}[x]/\left(p(x)\right)$ is a field, and calculate the dimension of R as a vector space over $\mathbf{Q}$.

8. Let M be a (left) module over a commutative ring R. The set

$$T(M) = \{m \in M \mid \exists x \in R \setminus \{0\}, xm = 0\}$$

is called the *torsion part* of M. Prove that, if R is an integral domain, then $T(M)$ is a submodule of M. Give an example to show that $T(M)$ may fail to be a submodule when R is not an integral domain.

9.Find a splitting field for $x^4 - 2x^2 - 6$ over $\mathbf{Q}$ and its degree.

10. Let $\mathbf{F} = \mathbf{Q}(\sqrt[3]{2}, \sqrt[3]{3}, \dots, \sqrt[3]{2007})$ be the field obtained from $\mathbf{Q}$ by joining all the numbers $\sqrt[3]{n}$, $2 \leq n \leq 2007$. Prove that the equation $x^5 + 9x + 6 = 0$ has no solutions in $\mathbf{F}$.